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Contents

Abstract

In this monograph we extend the classical algebraic theory of quadratic forms over fields to diagonal quadratic forms with invertible entries over broad classes of commutative, unitary rings where -1 is not a sum of squares and 2 is invertible. We accomplish this by:

(1) Extending the classical notion of matrix isometry of forms to a suitable notion of T-**isometry**, where T is a preorder of the given ring, A, or $T = A^2$.

(2) Introducing in this context three axioms expressing simple properties of (value) representation of elements of the ring by quadratic forms, well-known to hold in the field case.

Under these axioms we prove that the ring-theoretic approach based on T-isometry coincides with the formal approach formulated in terms of reduced special groups. This guarantees, for rings verifying these axioms, the validity of a number of important structural properties, notably the Arason-Pfister Hauptsatz, Milnor's mod 2 Witt ring conjecture, Marshall's signature conjecture, uniform upper bounds for the Pfister index of quadratic forms, a local-global Sylvester inertia law, etc. We call **(T)-faithfully quadratic** rings verifying these axioms.

A significant part of the monograph is devoted to prove quadratic faithfulness of certain outstanding (classes of) rings; among them, rings with many units satisfying a mild additional requirement, reduced f-rings (herein rings of continuous real-valued functions), and strictly representable rings.

Obviously, T-quadratic faithfulness depends on both the ring and the preorder T. We isolate a property of preorders defined solely in terms of the real spectrum of a given ring —that we baptise **unit-reflecting** preorders— which, for an extensive class of preordered rings, $\langle A, T \rangle$, turns out to be equivalent to the T-quadratic

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faithfulness of A. We show, e.g., that all preorders on the ring of continuous real-valued functions on a compact Hausdorff are unit-reflecting; we also give examples where this property fails.

Preface

This research monograph is an attempt to extend the classical algebraic theory of quadratic forms over fields − along the tradition initiated by E. Witt is the late 1930's and pursued by A. Pfister in the late 1960's − to the realm of (a broad class of) commutative, unitary rings. [1]

Obvious obstacles met at the outset of such an endeavour are:

- Quadratic forms over rings are seldom diagonalizable;
- Difficulties arise in handling forms with non-unit coefficients.

Recognizing these stumbling blocks, we consider only diagonal forms with unit coefficients.

It is a well established fact that the classical algebraic theory of quadratic forms yields some of its most appealing results when the underlying field is formally real (i.e., orderable). In this case, the notion of signature of a (diagonal) quadratic form at the various orders of the base field − a notion going back to Sylvester (1853) − draws an arithmetic bridge between the isometry of quadratic forms and the algebra of the field, embodied in Pfister's local-global principle.

Excluding from the outset the cases of rings of characteristic 2, by assuming that 2 is invertible, and of rings with imaginary elements, by assuming that -1 is not a square, the present monograph provides a unified treatment of quadratic form theory

- Over preordered rings (hereafter called **p-rings**), and
- In the case of square classes.[2]

In this setting, our approach is based on:

(1) The use, in the present context of our abstract, first-order axiomatic theory of quadratic forms, the theory of *special groups*, expounded in [**DM2**];[3]

(2) An extension of the classical notion of (matrix) isometry of forms to that of T-isometry, bringing into play the preorder T in the case of preordered rings (cf. 2.17; if B is a subset of a ring, write $B^{\times}$ for the units in B):

[1] For excellent expositions of the classical theory of quadratic forms over fields we recommend [**La**] and [**Sch**].

[2] A treatment of quadratic form theory over unitary, associative (but not necessarily commutative) rings in the broadest degree of generality is offered in Knus' book, [**Kn**]. However, this approach employs tools very different from ours.

[3] Corroborating a development envisaged by Manfred Knebusch at the outset of the theory of special groups.

(A) *Two n-dimensional quadratic forms $\varphi = \sum_{i=1}^n a_i X_i^2$, $\psi = \sum_{i=1}^n b_i X_i^2$, with a_i, $b_i \in A^\times$ are* T-isometric, *$\varphi \approx_T \psi$, if there is a sequence $\varphi_0, \varphi_1, \ldots, \varphi_k$ of n-dimensional diagonal forms over $A^\times$, so that $\varphi = \varphi_0$, $\psi = \varphi_k$ and for every $1 \leq i \leq k$, φ_i is* either *isometric to φ_{i-1} in the usual sense that there is a matrix $M \in \mathrm{GL}_n(A)$ such that $\varphi_i = M\varphi_{i-1}M^t$,* or *there are $t_1, \ldots, t_n \in T^\times$ such that $\varphi_i = \langle t_1x_1, \ldots, t_nx_n \rangle$ and $\varphi_{i-1} = \langle x_1, \ldots, x_n \rangle$.*

(B) To any p-ring, $\langle A, T \rangle$, we associate a structure $G_T(A)$, whose domain is $A^\times/T^\times$, endowed with the product operation induced by $A^\times$, together with a binary relation $\equiv_{G_T(A)}$, defined on ordered pairs of elements of $A^\times/T^\times$, called *binary isometry*, and having $-1 = -1/T^\times$ as a distinguished element. The structure $\langle G_T(A), \equiv_{G_T(A)}, -1 \rangle$ is not quite a special group in the sense of [**DM2**], Definition 1.2, p. 2, but it satisfies some of its axioms, constituting a *proto-special group* as in Definition 1.9 (see also Definition 6.3 of [**DM8**]). The ring-theoretic approach, based on the definition in (A), and the formal approach (via $G_T(A)$), though related, *are far from identical*.

(C) In Chapter 3 (see 3.1) we introduce three axioms, formulated in terms of T-isometry and the *value representation* relation D^v_T on $\langle A, T \rangle$ defined by: for a, $b_1, \ldots, b_n$ in $A^\times$

$$a \in D^v_T(b_1, \ldots, b_n) \quad \Leftrightarrow \quad \exists\, t_1, \ldots, t_n \in T \text{ such that } \quad a = \textstyle\sum_{i=1}^n t_i b_i.$$

These axioms express elementary properties of value representation, well-known in the classical theory of quadratic forms over fields. We then show (3.6) that, when satisfied by $\langle A, T \rangle$, these axioms are sufficient − and under mild assumptions, also necessary − to ensure identity between the ring-theoretic and formal approaches; in fact, we prove:

(C.*i*) The structure $G_T(A)$ is a special group.

(C.*ii*) T-isometry and value representation in $\langle A, T \rangle$ are faithfully coded by the corresponding formal notions in $G_T(A)$.

We call ***T*-faithfully quadratic** any p-ring $\langle A, T \rangle$ verifying these axioms. In fact, this setting, as well as the consequences $(C.i)$ and $(C.ii)$, apply, more generally, to forms with entries in certain subgroups of $A^\times$, called *T-subgroups*,[4] and also to the case where $T = A^2$; in this latter case, T-isometry is just matrix isometry. It also worth noticing that, under these axioms, an analog of the classical Pfister local-global principle holds for T-isometry (cf. Proposition 3.18 and Definition 3.14.(1)).

Quadratic faithfulness of $\langle A, A^2 \rangle$ ensures that the mod 2 K-theory of A obtained from that in [**Gu**] coincides with the K-theory of the special group $G(A)$ as defined in [**DM3**] and [**DM7**]. In fact, only the simplest of our axioms is needed here (2.16)

(D) In Chapter 4 we examine the behaviour of T-isometry under arbitrary products (Proposition 4.5) and prove preservation of T-quadratic faithfulness under such

[4] This generalization is not mere *ars gratia artis*; it plays a crucial role in the main result of [**DM9**].

products (Theorem 4.6). A number of basic results concerning the real spectrum of a ring, of later use in the text, are collected in section 2 of this chapter.

(E) With a view to applications of model-theoretic techniques, we discuss, in Chapter 5, the logical form of the axioms for T-quadratic faithfulness. These axioms are formalized by first-order sentences in the language of unitary rings (consisting of $+, \cdot\ , 0, 1, -1$) augmented by a unary predicate symbol for T (not needed if $T = A^2$). These sentences, given explicitly, are of a special form − known as *geometric* formulas − automatically guaranteeing preservation of T-quadratic faithfulness under (right-directed) inductive limits; this result, together with Theorem 4.6, yields preservation under reduced products which, by general model-theoretic considerations ([**CK**], Thm. 6.2.5′), implies the Horn axiomatizability of T-quadratic faithfulness. In the case $T = A^2$, we give explicit *Horn-geometric* axioms for quadratic faithfulness in the language of rings.

Chapters 6, 7 and 8 are devoted to prove that certain outstanding classes of p-rings, $\langle A, T \rangle$, are T-faithfully quadratic. The considerable effort demanded by some of these proofs (e.g., for the case of f-rings) is rewarded by the significance of the results thus reaped. These establish that the theory of diagonal quadratic forms with invertible entries over several classes of p-rings, important in mathematical practice, although far from being fields, possess many of the pleasant properties of quadratic form theory previously known to hold only in the case of formally real fields. Some of these results are gathered in Chapter 10.

(F) In Chapter 6 we deal with a large class of *rings with many units*. Using results due to Walter [**Wa**] we show that rings in this class are *completely faithfully quadratic*, i.e., T-faithfully quadratic for all preorders T and for $T = A^2$.

We observe that the class of rings with many units is axiomatizable by Horn-geometric first-order sentences in the language of unitary rings, guaranteeing that this class is closed under inductive limits and arbitrary reduced products. We also show that the ring of formal power series with coefficients in a ring with many units also has many units. To the best of the authors' knowledge these results are new.

(G) Next we consider, in Chapter 7, the class of p-rings $\langle A, T \rangle$ having the *bounded inversion* property, i.e., $1 + T \subseteq A^{\times}$. Generalizing, with a different proof, a result proved by Mahé [**Ma2**] for $T = \Sigma A^2$, we show that value representation in these rings has a strong property known as *transversality.*

(H) In Chapter 8 we turn to the study of quadratic form theory over *reduced f-rings*, i.e., subdirect products of linearly ordered integral domains. Rings of this type − of which the ring $\mathbb{C}(X)$ of continuous real-valued functions on a topological space X is an outstanding example − come endowed with a partial order that makes them into lattice-ordered rings. Our central result in Chapter 8 is Theorem 8.21, that reads:

> *If A is a reduced f-ring and T is a preorder of A containing the natural partial order ${T_\sharp}^A$ of A, then $\langle A, T \rangle$ is T-faithfully quadratic. Further, the reduced special group $G_T(A)$ associated to $\langle A, T \rangle$ (see (C.i) above) is a Boolean algebra, a quotient of the Boolean algebra of idempotents of A.*

The main steps in the proof of this result, Theorems 8.13 and 8.18, are also used elsewhere in the text.

In the important case where $A = \mathbb{C}(X)$ and $T = A^2$, this result implies *complete quadratic faithfulness* of all rings of continuous real-valued functions on any topological space X (Proposition 8.25.(b)).

If the space X is compact Hausdorff, we prove (Theorem 8.29) that *all* preorders of $\mathbb{C}(X)$ have, in addition, the property of being *unit-reflecting*, an important property considered in various parts of our text. For arbitrary reduced f-rings we only know that its natural partial order is unit-reflecting (Claim 2, proof of Theorem 8.20).

Among interesting classes of rings to which the previous results apply, are the *completely real function rings* and the *weakly real closed rings*. Some properties of these rings, as well as other examples, are considered in section 5 of Chapter 8.

(I) In Chapter 9 we examine the class of *strictly representable* rings, that is, p-rings $\langle A, T \rangle$ admitting a dense representation into $\mathbb{C}(X)$, for some compact Hausdorff space X, that, in addition, sends T into non-negative functions, and $T^{\times}$ consists of all the elements of A represented by strictly positive functions on X.

We show (Theorem 9.7) that these are exactly the bounded inversion p-rings $\langle A, T \rangle$, where the preorder T *is Archimedean*. As this equivalence shows, the notion of strict representability is intimately linked with the Becker-Schwartz representation theorem for Archimedean preorders (sometimes called the Kadison-Dubois Theorem). Outstanding examples are:

• The real holomorphy ring $H(K)$ of any formally real field K ([**Be3**], p. 21 ff), preordered by sums of squares, and

• For any p-ring $\langle A, T \rangle$, the convex hull of $\mathbb{Z}$ (with respect to T) in A, with the restricted preorder.

Once again, strictly representable p-rings $\langle A, T \rangle$ turn out to be T-faithfully quadratic (Theorem 9.9), and their associated special groups are Boolean algebras, namely the Boolean algebra $B(X)$ of clopens of the representing compact space X. In Theorem 9.13 we extend this result by showing, once more, that any preorder P of A containing T is unit-reflecting and $\langle A, P \rangle$ is P-faithfully quadratic.

(J) Finally, employing the Boolean-theoretic methods of [**DM2**] and the results of the papers [**DM3**], [**DM5**] and [**DM7**] we can harvest the fruits of the preceding efforts. In Chapter 10, we give a sample of results that can be thus obtained, e.g.:

(*J.i*) If A is an Archimedean and Pythagorean p-ring with weak bounded inversion, all mod 2 K-theoretic groups k_nA coincide with the Boolean algebra of clopens of the subspace of maximal points of Sper(A) (Theorem 10.2.(b)). In particular, Milnor's mod 2 Witt-ring conjecture holds for these rings (Theorem 10.2).

(*J.ii*) The Arason-Pfister Hauptsatz holds for arbitrary T-faithfully quadratic T-subgroups of any proper p-ring $\langle A, T \rangle$ (Theorem 10.4).

The results stated in (*J.iii*) $-$ (*J.iv*) below hold for p-rings, $\langle A, T \rangle$, where T is a preorder verifying either of the following requirements:

(1) T contains an Archimedean preorder P of A such that $\langle A, P \rangle$ has bounded inversion; or

(2) A is a reduced f-ring and T contains its natural partial order.

$(J.iii)$ For integers $n,\ m \geq 1$, the $(n,\ m)$-Pfister index of $\langle A, T \rangle$ (cf. [**DM5**]) is uniformly bounded above. That is, any quadratic form of dimension m and entries in $A^{\times}$ belonging to $I_T^n(A)$ (the n^{th}-power of the fundamental ideal of the Witt ring of A mod T) is Witt-equivalent to a linear combination (with unit coefficients) of at most max $\left\{1,\ \left[\dfrac{m}{2^n}\right]\right\}$ Pfister forms of degree n (Theorem 10.5.(c)).

$(J.iv)$ Marshall's signature conjecture holds − even in an improved form − for all natural signatures (cf. Definition 3.14) carried by $\langle A, T \rangle$; see Theorem 10.7 and Remark 10.9.

For reduced f-rings A, with its natural partial order denoted by $T_\sharp$, we prove the validity of

$(J.v)$ A (local-global) version of Sylvester's inertia law, giving a combinatorial characterization of $T_\sharp$-isometry (Theorem 10.10).

The symbol ■ indicates the end of a proof, of a statement or of an environment, while □ indicates the end of an assertion inside a proof (e.g., a Claim).

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CHAPTER 1

Basic Concepts

This chapter describes the notational conventions that will be used in all that follows, together with rather general facts on forms over an arbitrary non-empty set. The third section presents the concepts of proto-special, pre-special and special groups, together with their morphisms and some fundamental constructions.

1. Preliminaries

This section sets down the conventions and notations that will be adhered to consistently throughout the monograph.

1.1. We employ standard notation for the set-theoretic operations, calling attention to the following notational conventions:

a) $\bigcup A$ and $\bigcap A$ stand for the union and intersection of the elements of a *non-empty* set A. We set $\bigcup \emptyset = \emptyset$, while $\bigcap \emptyset$ is undefined.

b) If A, B are sets

$$A \setminus B \;=\; \{x \in A : x \notin B\} \qquad \text{and} \qquad A \triangle B \;=\; (A \setminus B) \cup (B \setminus A)$$

stand for their *difference* and *symmetric difference*, respectively.

c) $A \subseteq_f B$ stands for A is a *finite subset* of B.

d) If $f : A \longrightarrow B$ is a map, $D \subseteq A$ and $E \subseteq B$, $f[D]$ and $f^{-1}[E]$ stand for *image* and *inverse image* of D and E, respectively, by f. ■

1.2. a) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ denote the natural, integers, rational and real numbers, with their standard and well-known structures.

b) If $n \geq 1$ is an integer, we set $\underline{\boldsymbol{n}} = \{1, 2, \ldots, n\}$.

c) We write $\mathbb{Z}_2 = \{1, -1\}$ for the multiplicative group of units in $\mathbb{Z}$.

d) The two-element field will be denoted by $\mathbb{F}_2$. ■

1.3. Products and Disjoint Unions. If $\{A_i : i \in I\}$ is a family of sets,

$$\prod_{i \in I} A_i \;=\; \{\langle a_i \rangle_{i \in I} : \forall\, i \in I,\, a_i \in A_i\}$$
$$\coprod_{i \in I} A_i \;=\; \bigcup_{i \in I} \{i\} \times A_i,$$

indicate their *product* and *disjoint union*, respectively.

As usual, if $A_i = A$ for all $i \in I$, we write A^I for their product. In particular, if $n \geq 1$ is an integer,

$$A^n \;=\; \underbrace{A \times A \times \cdots \times A}_{n} \;=\; \{\langle a_1, \ldots, a_n \rangle : a_i \in A,\, i \in \underline{\boldsymbol{n}}\}.$$

If the set I is clear from context, we write $\overline{a} = \langle a_i \rangle_{i \in I}$ for an element of $\prod_{i \in I} A_i$. For each $j \in I$, there is a canonical *coordinate projection*

$$p_j : \prod_{i \in I} A_i \longrightarrow A_j, \text{ given by } p_j(\overline{a}) = a_j.$$

Dually, there is, for each $j \in I$, a natural map

$$\eta_j\colon A_j \longrightarrow \coprod_{i \in I} A_i, \text{ given by } \eta_j(z) = \langle j, z \rangle.$$

If $f_i : A_i \longrightarrow B_i$, $i \in I$, is a family of maps, define

$$\prod_{i \in I} f_i : \prod_{i \in I} A_i \longrightarrow \prod_{i \in I} B_i, \text{ given by } \prod_{i \in I} f_i(\overline{a}) = (f_i(a_i))_{i \in I},$$

called the *product* of the f_i. If $f : A \longrightarrow B$ is a map and I is a set, we write $f(\overline{a})$ for $f^I(\overline{a})$, where $f^I : A^I \longrightarrow B^I$ is the power map. ■

2. Geometric and Horn-Geometric Theories

We assume familiarity with first-order languages, their structures and morphisms. Standard references are [**CK**] and [**Ho**]. For the reader's benefit, we recall some basic definitions and results.

DEFINITION 1.4. *Let L be a first-order language with equality. If A is a L-structure and T is a constant, function or relation symbol in L, write T^A for its interpretation in A.*

Let A, B be L-structures, let $f : A \longrightarrow B$ be a map and let $\varphi(v_1, \ldots, v_n)$ be a formula of L in the free variables $\overline{v} = \langle v_1, \ldots, v_n \rangle$. For $\overline{a} = \langle a_1, \ldots, a_n \rangle \in A^n$, write $f(\overline{a})$ for $\langle f(a_1), \ldots, f(a_n) \rangle \in B^n$.

a) f **preserves** *φ if for all $\overline{a} \in A^n$, $A \models \varphi[\overline{a}] \Rightarrow B \models \varphi[f(\overline{a})]$; f* **reflects** *φ if the reverse implication holds.*

b) f is a ***L*-morphism** *if it preserves all atomic formulas and a* ***L*-embedding** *if it preserves and reflects all atomic formulas.*

c) Let ***L*-str** *be the category of L-structures and L-morphisms. If Σ is a set of sentences in L, write* **Σ-mod** *for the subcategory of* ***L*-str** *whose objects are the models of Σ.*

d) Assume that $f : A \longrightarrow B$ is a L-morphism. A is **positively existentially closed** *in B* **along** *f if f reflects all positive existential L-formulas. Such L-morphisms are also called* **pure**. *Whenever A is a substructure of B and f is the inclusion, we say that A is* **positively existentially closed in** *B.*

e) A formula of L, in the free variables $\overline{t}$, is **geometrical** *if it is the negation of an atomic formula* **or** *of the form*

$$\forall \overline{v}\ (\exists \overline{u}\ \varphi_1(\overline{v}; \overline{u}; \overline{t}) \longrightarrow \exists \overline{w}\ \varphi_2(\overline{v}; \overline{w}; \overline{t})),$$

where φ_1, φ_2 are positive and quantifier-free. A **geometrical theory** *in L is a theory possessing a set of geometrical axioms.*

f) A formula in L is **positive primitive (pp-) formula** *if it is of the form $\exists \overline{v}\, \varphi(\overline{v}; \overline{t})$, where φ is a conjunction of atomic formulas.*

g) A formula in L is **Horn-geometrical** *if it is the negation of an atomic formula* **or** *of the form $\forall \overline{v}\ (\varphi \longrightarrow \psi)$, where φ and ψ are pp-formulas. A* **Horn-geometrical**

theory *in L is a theory possessing a set of Horn-geometrical axioms. Clearly, any Horn-geometrical formula is geometrical.* ■

Remarks 1.5. For the logical arguments corroborating the assertions below the reader may consult, for example, section 10 of chapter 2 in [**Men**].

a) Any conjunction of pp-formulas is logically equivalent to a pp-formula.

b) Any L-formula of the form $\exists \overline{z}\ \varphi(\overline{z})$, where φ is positive and quantifier-free (i.e., positive-existential) is logically equivalent to a disjunction of pp-formulas.

c) Let $\varphi(\overline{z})$ be an L-formula of the form $\psi_0(\overline{z}) \wedge \bigwedge_{j=1}^{p} \exists \overline{v}_j \psi_j(\overline{v}_j;\, \overline{z})$, where ψ_0 is quantifier free. We may assume that the variables $\overline{v}_j$, $1 \leq j \leq p$, are all distinct, and distinct from the $\overline{z}$. Then, φ is logically equivalent to

$$\exists \overline{v}_1\ \exists \overline{v}_2 \cdots \exists \overline{v}_p\ (\psi_0(\overline{z}) \wedge \bigwedge_{j=1}^{p} \psi_j(\overline{v}_j;\, \overline{z})).$$

In particular, if every ψ_j $(0 \leq j \leq p)$ is a conjunction of atomic formulas, then $\varphi(\overline{z})$ is logically equivalent to a pp-formula.

d) Let $\varphi(\overline{z}) = \forall \overline{v}\ (\exists \overline{x}\ \varphi_1(\overline{x},\, \overline{v};\, \overline{z}) \to \exists\ \overline{y}\ (\varphi_2(\overline{y},\, \overline{v};\, \overline{z})))$, where φ_i, $i = 1, 2$, are conjunctions of atomic formulas, be a Horn-geometrical L-formula. We may assume that the variables $\overline{x}$, $\overline{y}$, $\overline{z}$ are all distinct. Then, $\varphi(\overline{z})$ is logically equivalent to

$$\text{(I)} \qquad \forall \overline{v}\ \forall \overline{x}\ \exists \overline{y}\ (\varphi_1(\overline{x},\, \overline{v};\, \overline{z}) \to \varphi_2(\overline{y},\, \overline{v};\, \overline{z})).$$

If $\varphi_2 = \bigwedge_{k=1}^{q} \psi_k$, where ψ_k are atomic L-formulas, then, $\varphi_1 \to \varphi_2$ is logically equivalent to

$$\text{(II)} \qquad \bigwedge_{k=1}^{q} (\varphi_1 \to \psi_k).$$

It follows from (I) and (II) that $\varphi(\overline{z})$ is logically equivalent to

$$\forall \overline{v}\ \forall \overline{x}\ \exists \overline{y} \bigwedge_{k=1}^{q} (\varphi_1 \to \psi_k),$$

and φ is a **Horn formula** according to the definition in the paragraph preceding Proposition 6.2.2 of [**CK**]. We also note that the negation of an atomic L-formula is a Horn formula. This explains our choice of the term *Horn-geometrical*. ■

The following is straightforward:

Lemma 1.6. *Let $f : A \longrightarrow B$ be a L-morphism and assume that A is positively existentially closed in B along f. If Σ is geometrical L-theory, then $B \models \Sigma$ implies $A \models \Sigma$. In particular, the same holds if Σ is a Horn-geometrical L-theory.* ■

3. Forms over Sets

The concepts defined below will be fundamental in all that follows:

Definition 1.7. *Let A, $B \neq \emptyset$ be sets and let $n \geq 1$ be an integer.*

*a) A **n-form over** A is an element of A^n. If φ is a n-form over A, its **dimension**, $\dim(\varphi)$, is n.*

b) If $A \xrightarrow{f} B$ is a map and $\varphi = \langle a_1, \ldots, a_n \rangle$ is a n-form over A, write $f \star \varphi = \langle f(a_1), \ldots, f(a_n) \rangle$ for the image n-form over B.

c) If $\varphi = \langle u_1, \ldots, u_n \rangle$, $\psi = \langle v_1, \ldots, v_m \rangle$ are forms over A, then

$$\varphi \oplus \psi = \langle u_1, \ldots, u_n, v_1, \ldots, v_m \rangle$$

*is the **direct sum** of φ, ψ.*

d) Let $\equiv$ be a binary relation on A^2. We define a sequence $\equiv_n$ of binary relations on A^n, $n \geq 1$, as follows:

$(d.1)$ $\equiv_1 \;=\; \{\langle a, a\rangle \in A^2 : a \in A\}$; $(d.2)$ $\equiv_2 \;=\; \equiv$;

$(d.3)$ *For $n \geq 3$, we proceed by induction: $\langle a_1, \ldots, a_n\rangle \equiv_n \langle b_1, \ldots, b_n\rangle$ iff there are x, y, z_3, $\ldots$, z_n in A such that*

(i) $\langle a_1, x\rangle \equiv_2 \langle b_1, y\rangle$; (ii) $\langle a_2, \ldots, a_n\rangle \equiv_{n-1} \langle x, z_3, \ldots, z_n\rangle$;

(iii) $\langle b_2, \ldots, b_n\rangle \equiv_{n-1} \langle y, z_3, \ldots, z_n\rangle$.

We shall frequently abuse notation and let $\equiv$ also stand for its extension to A^n, $n \geq 3$. ■

Lemma 1.8. *Let $A \neq \emptyset$ be a set and let $\equiv$ be a binary relation on A^2. Let $n, m \geq 1$ be integers.*

a) The direct sum of forms over A is associative.

b) If $\equiv$ is reflexive on A^2, then for all $n \geq 2$, its extension to A^n is reflexive.

c) If $\equiv$ is symmetric on A^2, then for all $n \geq 2$, its extension to A^n is symmetric.

d) Assume that $\equiv$ is reflexive and symmetric on A^2. If φ_1, φ_2 are n-forms over A and ψ_1, ψ_2 are m-forms over A, then

$$\varphi_1 \equiv \varphi_2 \quad \text{and} \quad \psi_1 \equiv \psi_2 \quad \Rightarrow \quad \varphi_1 \oplus \psi_1 \equiv \varphi_2 \oplus \psi_2.$$

e) Assume $\equiv$ is reflexive on A^2 and that for all u, $v \in A$ we have $\langle u, v\rangle \equiv \langle v, u\rangle$. Then, for all a, x, $y \in A$, $\langle a, x, y\rangle \equiv \langle a, y, x\rangle$.

f) With notation as in 1.7.(d), the following are equivalent:

(1) *For all $n \geq 2$, $\equiv_n$ is an equivalence relation;*

(2) *$\equiv$ is an equivalence relation on A^2 and $\equiv_3$ is transitive.*

Proof. Item (a) is clear. For items (b) and (c), let $n \geq 3$. Then:

• If $\equiv$ is reflexive on A^2 and $\varphi = \langle a_1, \ldots, a_n\rangle$, with notation as in 1.7.(d), just take $x = a_2 = y$ and $z_k = a_k$, if $k \geq 3$;

• Assume $\equiv$ is symmetric and that x, y, z_3, $\ldots$, $z_n \in A$ are witnesses that $\langle a_1, \ldots, a_n\rangle \equiv_n \langle b_1, \ldots, b_n\rangle$, as in 1.7.(d.3); it is straightforward from the definition of $\equiv_n$ that those same elements of A are witnesses for $\langle b_1, \ldots, b_n\rangle \equiv_n \langle a_1, \ldots, a_n\rangle$.

d) We prove the statement by induction on $n = \dim(\varphi_i)$, $i = 1, 2$. For $n = 1$, we have $\varphi_1 = \varphi_2 = \langle a\rangle$, $a \in A$. Let $\psi_1 = \langle c_1, \ldots, c_m\rangle$, $\psi_2 = \langle d_1, \ldots, d_m\rangle$; then, items (c) and (d) yield the relations

$$\langle a, c_1\rangle \equiv \langle a, c_1\rangle, \quad \psi_1 \equiv \psi_1 \quad \text{and} \quad \psi_2 \equiv \psi_1,$$

showing that $\langle a\rangle \oplus \psi_1 \equiv \langle a\rangle \oplus \psi_2$. Assume the result true for forms of dimension n in the first term of the direct sum, and suppose $\varphi_1 = \langle a, a_1, \ldots, a_n\rangle$, $\varphi_2 = \langle b, b_1, \ldots, b_n\rangle$. Let x, y, $\theta = \langle z_2, \ldots, z_n\rangle$ be witnesses to $\varphi_1 \equiv \varphi_2$, i.e.,

$$(*)\qquad \begin{cases} \langle a, x\rangle \equiv \langle b, y\rangle, \;\; \langle a_1, \ldots, a_n\rangle \equiv \langle x\rangle \oplus \theta \\ \text{and} \\ \langle b_1, \ldots, b_n\rangle \equiv \langle y\rangle \oplus \theta. \end{cases}$$

The relations in (*) and the induction hypothesis give

$$\begin{cases} \langle a_1, \ldots, a_n \rangle \oplus \psi_1 \equiv \langle x \rangle \oplus \theta \oplus \psi_1 \\ \qquad\text{and} \\ \langle b_1, \ldots, b_n \rangle \oplus \psi_2 \equiv \langle y \rangle \oplus \theta \oplus \psi_1, \end{cases}$$

that together with the first relation in (*) yield $\varphi_1 \oplus \psi_1 \equiv \varphi_2 \oplus \psi_2$, completing the induction step.

e) The assumptions and the relations

$$\langle a, x \rangle \equiv \langle a, x \rangle, \quad \langle x, y \rangle \equiv \langle x, y \rangle \quad \text{and} \quad \langle y, x \rangle \equiv \langle x, y \rangle$$

guarantee that $\langle a, x, y \rangle \equiv \langle a, y, x \rangle$.

f) Since $\equiv_2 = \equiv$ and $\equiv_1$ is the diagonal of A^2, it is clear that (1) $\Rightarrow$ (2). For the converse, in view of items (b) and (c), it remains to check that if $\equiv_3$ is transitive, the same is true of $\equiv_n$ for all $n \geq 4$. It is straightforward that with the statements established above, the proof of (1) $\Rightarrow$ (2) in Theorem 1.23 of [**DM2**] (pp. 16–17), applies as it stands to establish (f), ending the proof. ∎

4. Proto-Special Groups and Special Groups

The basic reference on the theory of special groups is [**DM2**]. However, in the course of development of the theory of special groups and its applications it became clear that it was profitable to have a more general concept, also first-order describable, and functorially associated to very wide classes of rings. The next definition was originally introduced in [**DM8**].

Definition 1.9. *a*) *A* **proto-special group (π-SG)**, *is a triple,* $G = \langle G, \equiv_G, -1 \rangle$, *consisting of*

- *A group, G, of exponent two, written multiplicatively* [1]*;*
- *A distinguished element,* -1, *in G;* (*write* $-x$ *for* $-1 \cdot x$, $\forall\, x \in G$)*;*
- *A binary relation* $\equiv_G$ *on* $G \times G$, *called* **isometry**, *satisfying the following axioms:*[2] *for all* $x, a, b, c, d \in G$

[SG 0] : $\equiv_G$ *is an equivalence relation;*

[SG 1] : $\langle a, b \rangle \equiv_G \langle b, a \rangle$*;* [SG 2] : $\langle a, -a \rangle \equiv_G \langle 1, -1 \rangle$*;*

[SG 3] : $\langle a, b \rangle \equiv_G \langle c, d \rangle \;\Rightarrow\; ab = cd$*;*

[SG 5] : $\langle a, b \rangle \equiv_G \langle c, d \rangle \;\Rightarrow\; \langle xa, xb \rangle \equiv_G \langle xc, xd \rangle$.

G is **reduced** (abbreviated π-RSG) *if* $1 \neq -1$ *and it satisfies*

[red] : $\langle a, a \rangle \equiv_G \langle 1, 1 \rangle \;\Rightarrow\; a = 1$.

b) *A π-SG is a* **pre-special group (pSG)** *if it satisfies*

[SG 4] : *For all* $a, b, c, d \in G$,

$$\langle a, b \rangle \equiv_G \langle c, d \rangle \;\Rightarrow\; \langle a, -c \rangle \equiv_G \langle -b, d \rangle.$$

[1] $x^2 = 1$, where 1 is its identity.

[2] Numbered as in Definition 1.2 (p. 2) of [**DM2**].

c) A π-SG is a **special group (SG)** *if it is a pSG verifying*

[SG 6] (3-transitivity): *The extension of $\equiv_G$ to G^3, cf.* 1.7.(*d*), *is a transitive relation.*

d) As in 1.7.(*a*), *a* ***n*-form over** *G is an element of G^n. If $\varphi = \langle a_1, \ldots, a_n \rangle$ is a form over G and $c \in G$, write $c\varphi$ for $\langle ca_1, \ldots, ca_n \rangle$. The product $a_1 \cdot \cdots \cdot a_n$ is the* **discriminant of φ**, *written $d(\varphi)$. Two n-forms over G, φ, ψ, are* **isometric** *if $\varphi \equiv_G \psi$.*

e) If $\varphi = \langle a_1, \ldots, a_n \rangle$ and $\psi = \langle b_1, \ldots, b_m \rangle$ are forms over G,

$$\varphi \otimes \psi \;=\; \langle a_1b_1, a_1b_2, \ldots, a_1b_m, \ldots, a_nb_1, \ldots a_nb_m \rangle \;=\; \bigoplus_{i=1}^{n} a_i\psi$$

is the **tensor product** *of φ and ψ.*

f) A **Pfister form over** *G is a form $\mathcal{P}$ isometric to $\bigotimes_{i=1}^{n} \langle 1, a_i \rangle$, where $a_i \in G$, $i \in \underline{\boldsymbol{n}}$. The integer n is the* **degree** *of $\mathcal{P}$; note that* $\dim(\mathcal{P}) = 2^n$.

g) If $G = \langle G, \equiv_G, -1 \rangle$ is a π-SG and φ is a n-form over G,

$$D_G(\varphi) = \{z \in G : \exists\, z_2, \ldots, z_n \in G \text{ such that } \langle z, z_2, \ldots, z_n \rangle \equiv_G \varphi\},$$

is the set of elements **represented by** *φ in G.* ■

Remarks 1.10. *a*) The first-order language of special groups, L_{SG} has, besides equality and a function symbol for multiplication, a quaternary relation for *binary isometry*, together with constants 1 (the group identity) and -1: $L_{SG} = \{=, \cdot, \equiv, 1, -1\}$.

It is clear that proto-special groups, pre-special groups, special groups and their reduced counterparts are Horn-geometrical theories in the language L_{SG} according to Definition 1.4.(g).

b) Let G be a π-SG. Since G has exponent two ($x^2 = 1$, for all x),

(*i*) By [SG 3], $\langle z, u \rangle \equiv_G \langle x, y \rangle$ entails $u = zxy$;

(*ii*) [SG 0] and [SG 1] imply $\{x, y\} \subseteq D_G(x, y)$.

c) In Chapter 1 of [**DM2**] the reader will find many examples of special groups. Interesting examples of proto and pre-special groups will appear in Chapter 2 below.

d) It is easily verified, using [SG 5], that a π-SG G is *reduced* iff

$$\text{For all } x, a \in G, \quad x \in D_G(a, a) \;\Rightarrow\; x = a.$$

e) The group of units of the integers, $\mathbb{Z}_2$, has a natural structure of *reduced special group*, where for a, b, c, $d \in \mathbb{Z}_2$,

$$\langle a, b \rangle \equiv_{\mathbb{Z}_2} \langle c, d \rangle \quad \text{iff} \quad a + b = c + d,$$

where the sum is computed in $\mathbb{Z}$. We have $D_{\mathbb{Z}_2}(1, 1) = \{1\}$ and $D_{\mathbb{Z}_2}(1, -1) = \mathbb{Z}_2$. ■

Definition 1.11. *A subgroup Δ of a π-SG, G, is* **saturated** *if for all $a \in G$, $a \in \Delta \;\Rightarrow\; D_G(1, a) \subseteq \Delta$.* ■

It is straightforward that arbitrary intersections and up-directed unions of saturated subgroups of a π-SG are again saturated. In Chapter 2 of [**DM2**] the reader will find an account of saturated subgroups of special groups and related topics, such as Pfister subgroups and quotients.

DEFINITION 1.12. *a) If* $G_i = \langle G_i, \equiv_{G_i}, -1 \rangle$ *are* π*-SGs,* $i = 1, 2$*, a* **morphism of π-SGs***,* $h : G_1 \longrightarrow G_2$*, is a* L_{SG}*-morphism, i.e., a morphism of the underlying groups, such that* $h(-1) = -1$ *and* $\forall\ a, b, c, d \in G_1$*,*

$$\langle a, b \rangle \equiv_{G_1} \langle c, d \rangle \quad \Rightarrow \quad \langle h(a), h(b) \rangle \equiv_{G_2} \langle h(c), h(d) \rangle.$$

Write **π-SG** *and* **π-RSG** *for the categories of* π*-SGs and* π*-RSGs, respectively. Similarly, one defines the categories* **pSG***,* **SG***,* **pRSG** *and* **RSG** *of pre-special groups, of special groups and their reduced counterparts, respectively. The notion of morphism in each of these categories is the same as that of proto-special groups.*

b) (Compare with Definition 3.1 and Notation 3.5, pp. 50 - 51, [**DM2**]) *Let* G *be a* π*-special group.*

(1) *A* **SG-character** *of* G (*or, simply, a* **character** *of* G) *is a is* π*-SG morphism from* G *to* $\mathbb{Z}_2$ (cf. 1.10.(d)). *Write* X_G *for the* (*possibly empty*) *set of characters of* G*.*

(2) *If* $\varphi = \langle x_1, \ldots, x_n \rangle$ *is a form over* G *and* $\sigma \in X_G$*, the integer* $sgn_\sigma(\varphi) = \sum_{i=1}^n \sigma(x_i)$ *is the* **signature** *of* φ **at** σ*.* ■

The next result generalizes to π-SGs statements proven in [**DM2**] for pre-special groups.

LEMMA 1.13. *Let* G *be a proto-special group.*

a) If a, b, c, d *are elements of* G*, then* $\langle a, b \rangle \equiv_G \langle c, d \rangle$ *iff* $ab = cd$ *and* $ac \in D_G(1, cd)$*. Moreover,*

$$D_G(1, a) \;=\; \{z \in G : z\langle 1, a \rangle \equiv_G \langle 1, a \rangle\}.$$

In particular, $D_G(1, a)$ *is a* **subgroup** *of* G*.*

b) If φ*,* ψ *are* n*-forms over* G*, then* $\varphi \equiv_G \psi$ *implies* $d(\varphi) = d(\psi)$*.*

c) The direct sum and the tensor product of isometric forms are isometric.

d) Let G*,* H *be* π*-SGs and let* φ*,* ψ *be* n*-forms over* G*.*

(1) *A map* $G \xrightarrow{f} H$ *is a* π*-SG morphism iff it is a group morphism taking* -1 *to* -1 *and satisfying*

$$\text{For all } x, y \in G, \quad x \in D_G(1, y) \;\Rightarrow\; f(x) \in D_H(1, f(y)).$$

(2) *If* $G \xrightarrow{f} H$ *is a morphism of* π*-SGs, then* $\varphi \equiv_G \psi$ *implies* $f \star \varphi \equiv_H f \star \psi$*.*

(3) *If* $G \xrightarrow{f} H$ *is* π*-SG morphism and* H *is reduced, then* $\ker f := f^{-1}[1]$ *is a proper saturated subgroup* (*cf.* 1.11) *of* G*.*

(4) *A map* $G \xrightarrow{\sigma} \mathbb{Z}_2$ *is a SG-character of* G *iff it is a group morphism taking* -1 *to* -1 *and whose kernel is a proper saturated subgroup of* G*. Moreover, if* $f : G \longrightarrow H$ *is a* π*-SG morphism and* $\sigma \in X_H$*, then* $\sigma \circ f \in X_G$*.*

PROOF. The proof of Lemma 1.5 in [**DM2**] (p. 3) applies *ipsis litteris* to yield items (a) and (b).

c) The statement concerning direct sum follows from Lemma 1.8.(d). Regarding tensor products, first note that if φ, ψ are forms of the same dimension over G and $a \in G$, then

$$\text{(I)} \qquad \varphi \equiv_G \psi \;\;\Rightarrow\;\; a\varphi \equiv_G a\psi.$$

Indeed, if $\dim(\varphi) = \dim(\psi) = 2$, (I) follows from [SG 5], while the general case is easily obtained by induction. Now let $\varphi_1 = \langle a_1, \ldots, a_n \rangle \equiv_G \varphi_2$ and $\psi_1 \equiv_G \psi_2 = \langle b_1, \ldots, b_m \rangle$. Then, using (I) and the preservation of isometry by direct sums, we obtain

$$\begin{aligned} \varphi_1 \otimes \psi_1 &= \textstyle\bigoplus_{k=1}^{n} a_k\psi_1 \;\equiv_G\; \bigoplus_{k=1}^{n} a_k\psi_2 \;=\; \varphi_1 \otimes \psi_2 \\ &= \textstyle\bigoplus_{j=1}^{m} b_j\varphi_1 \;\equiv_G\; \bigoplus_{j=1}^{m} b_j\varphi_2 \;=\; \varphi_2 \otimes \psi_2. \end{aligned}$$

Parts (1), (2) and (4) of item (d) can be established exactly as in the proof of Lemma 1.12 of [**DM2**] (p. 11). Regarding (3), since $1 \neq -1$ in H (it is reduced), $-1 \notin \ker f$ and so it is a proper subgroup of G. To show $\ker f$ is saturated, let $a \in \ker f$ and $b \in D_G(1, a)$; by (d.1), this implies $f(b) \in D_H(1, f(a)) = D_H(1, 1)$, and hence, since H is reduced, we conclude $f(b) = 1$, i.e., $b \in \ker f$, as needed. ■

DEFINITION 1.14. *Let $f : G \longrightarrow H$ be a morphism of π-SGs. We say that*

a) *f is* **complete** *if for all forms φ, ψ of the same dimension over G,*

$$\varphi \equiv_G \psi \quad\Leftrightarrow\quad f \star \varphi \equiv_H f \star \psi.$$

b) *f is a* **SG-embedding** *if it is injective and*

$$(*) \qquad \text{For all } a, b \in G, \;\; a \in D_G(1, b) \;\Leftrightarrow\; f(a) \in D_H(1, f(b)).$$

c) *f is a* **complete embedding** *if it is a complete SG-embedding.* ■

REMARK 1.15. If $f : G \longrightarrow H$ is a morphism of π-SGs and G is *reduced*, then property (*) in 1.14.(b) entails injectivity. Indeed, if $f(a) = 1$ for $a \in G$, then $f(a) = 1 \in D_H(1, f(1))$ yields $a \in D_G(1, 1)$; since G is reduced, $a = 1$. Hence, $\ker f = \{1\}$ and f is injective. In particular, if G is reduced, any complete SG-morphism is a complete embedding. ■

CHAPTER 2

Rings and Special Groups

In this Chapter, after setting down notational conventions and some fundamental notions in section 1, we describe and explore some of the basic concepts in the work to follow.

Let A be a unital commutative ring in which 2 is a unit and let $T = A^2$ or a proper preorder of A. Let $A^{\times}$ be the group of units in A.

In section 2 we construct a proto-special group, $G_T(S)$, associated to each pair $\langle S, T \rangle$, where S is a subgroup of $A^{\times}$, containing $T^{\times}$, (called a *T-subgroup of A*). This construction is, in fact, a fully geometrical functor, i.e., preserves arbitrary products and right-directed colimits.

In section 3, we extend the K-theory of special groups, introduced in [**DM3**], to proto-special groups, in order to show, in case $T = A^2$ and A satisfies a certain transversality condition for binary forms, that (the ring-theoretic version of) Milnor's mod 2 K-theory of A and the K-theory of the π-SG associated to the pair $\langle A, A^2 \rangle$ are naturally isomorphic.

In section 4, we introduce and study a ring-theoretic notion of T-isometry, that will play a crucial role in the sequel. Two important observations about this notion are:

- In case $T = A^2$, T-isometry reduces to standard matrix isometry (cf. Remark 2.18).
- When T is a preorder of a field, k, two forms over k are T-isometric iff they have the same signature at every order of k containing T.

We then introduce several notions of representation (2.24), developing their basic properties.

1. Conventions, Notation and Fundamental Concepts

The theory developed in these notes applies to the cases of a given preorder of a ring A, of sums of squares in A *and* of the set of squares A^2. Many proofs are entirely similar in all cases. To avoid needless repetition, we have set our notation in such a way that the cases can be treated simultaneously, while being able to distinguish among these three situations, when necessary.

2.1. a) In all that follows, the word **ring** stands for **commutative unitary ring in which 2 is a unit**. If A is a ring, $D \subseteq A$ and $x \in A$, we set

$$A^{\times} = \text{group of units of } A; \qquad D^{\times} = D \cap A^{\times};$$
$$D^2 = \{d^2 \in A : d \in D\}; \qquad xD = \{xd : d \in D\};$$

$\Sigma D^2 \;=\; \{\sum_{i=1}^{n} d_i^2 : \{d_1, \ldots, d_n\} \subseteq D \text{ and } n \in \mathbb{N},\, n \geq 1\}$.

$\mathrm{GL}_n(A)$ is the ring of invertible square matrices of size $n \geq 1$.

b) Let A be a ring. A subset $T \subseteq A$ is a **preorder** on A if T is closed under sums, products and contains A^2; T is **proper** if $T \neq A$. Since $2 \in A^\times$, a preorder T on A is proper iff $-1 \notin T$.

c) A proper **p-ring (preordered ring)**, $\langle A, T \rangle$, is a ring A with a proper preorder T on A (i.e., $-1 \notin T$). Clearly, $T^\times$ is subgroup of the multiplicative group $A^\times$, and A is **semi-real** ($-1 \notin \Sigma A^2$, since $\Sigma A^2 \subseteq T$).

d) The language of p-rings is the first-order language of rings, augmented by a unary predicate to be interpreted as a preorder. Clearly, the theory of p-rings is Horn-geometrical (cf. 1.4.(g)) in this language.

e) A **morphism of p-rings**, $f : \langle A_1, T_1 \rangle \longrightarrow \langle A_2, T_2 \rangle$ is a morphism of unitary rings such that $f[T_1] \subseteq T_2$. Write **p-Ring** for the category of p-rings and their morphisms.

Warning. Unless explicitly stated, *all* p-rings are proper. Recall our blanket assumption that in the rings here considered, -1 is not a square. ■

DEFINITION 2.2. *Let T be a preorder of a ring A, or $T = A^2$. A **T-subgroup** of A is a subset S of $A^\times$ containing $\{-1\} \cup T^\times$ and closed under products. The A^2-subgroups of A will be called **q-subgroups** ("q" for "quadratic").* ■

2.3. **Examples and Remarks** . Let A be a ring and let T be a proper preorder of A, or $T = A^2$.

a) If S is a T-subgroup of A, then S is a q-subgroup of $A^\times$, and a subgroup of $A^\times$, because $T^\times$ contains all invertible squares.

b) The smallest T-subgroup of A is $T^\times \cup -T^\times$; the largest is, of course, $A^\times$ itself.

c) The family of T-subgroups of A is closed under arbitrary intersections. Moreover, the union of any up-directed family of T-subgroups is a T-subgroup. ■

2.4. **Diagonal S-quadratic forms in free A-modules.** Let $n \geq 1$ be an integer and let A^n be the free n-dimensional A-module. Let S be a q-subgroup of A.

2.4.(a) To $\langle a_1, \ldots, a_n \rangle \in S^n$, we associate:

(1) The diagonal quadratic form, $\sum_{i=1}^{n} a_i X_i^2$, which, as usual, we denote by $\langle a_1, \ldots, a_n \rangle$.

(2) The diagonal matrix in $\mathrm{GL}_n(A)$, $\mathscr{M}(a_1, \ldots, a_n)$, whose non-zero entries are precisely $a_1, \ldots, a_n$ (i.e., the $\langle k, k \rangle$-entry of $\mathscr{M}$ is a_k).

(3) The discriminant of $\langle a_1, \ldots, a_n \rangle$ is the unit $a_1 a_2 \cdot \ldots \cdot a_n$ of S, exactly the determinant of $\mathscr{M}(a_1, \ldots, a_n)$.

Whenever $\varphi = \langle a_1, \ldots, a_n \rangle$ is a n-form over S, then $\mathscr{M}(\varphi)$ will stand for $\mathscr{M}(a_1, \ldots, a_n)$, while $d(\varphi)$ is the discriminant of φ.

2.4.(b) If $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_n \rangle$ are n-forms over S, define

$$\varphi \approx \psi \quad \text{iff} \quad \exists\, M \in \mathrm{GL}_n(A) \text{ such that } M\mathscr{M}(\varphi)M^t \;=\; \mathscr{M}(\psi).$$

Note that $d(\varphi)\, det(M)^2 = d(\psi)$, where $det\, M$ (or $det(M)$) is the determinant of M. The relation $\approx$, called **matrix isometry**, is an equivalence relation.

2.4.(c) If $\langle\, a_1, \ldots, a_n \,\rangle \in S^n$, $\langle\, c_1, \ldots, c_n \,\rangle \in A^{\times n}$ and σ is a permutation of $\{1, \ldots, n\}$, then, just as in Lemma 1.28 in [**DM6**] we have

(1) $\langle\, a_1, \ldots, a_n \,\rangle \;\approx\; \langle\, c_1^2 a_1, \ldots, c_n^2 a_n \,\rangle$;

(2) $\langle\, a_1, \ldots, a_n \,\rangle \;\approx\; \langle\, a_{\sigma(1)}, \ldots, a_{\sigma(n)} \,\rangle$.

2.4.(d) As usual, if φ, ψ, θ are forms over S, then

$$\varphi \approx \psi \quad \Rightarrow \quad \varphi \oplus \theta \approx \psi \oplus \theta \;\text{ and }\; \varphi \otimes \theta \approx \psi \otimes \theta. \qquad \blacksquare$$

2. Rings and Proto-Special Groups

2.5. **Binary Representation.** Let A be a ring and let T be a preorder of A, or $T = A^2$ and let S be a T-subgroup of A. Let

$$G_T(S) = S/T^{\times} \quad\text{and}\quad q_T : S \longrightarrow G_T(S)$$

be the quotient group and canonical projection, respectively; to ease notation, write a^T for $q_T(a)$. Thus, for $a, b \in S$,

$$(*) \qquad a^T = b^T \quad \Leftrightarrow \quad ab \in T^{\times} \quad \Leftrightarrow \quad \exists\, t \in T^{\times} \text{ such that } b = at,$$

and $G_T(S) \;=\; \{a^T : a \in S\}$. With a slight abuse of notation, we write 1 and -1 for 1^T, $(-1)^T \in G_T(S)$, respectively.

Since $A^2 \subseteq T$, $G_T(S)$ is a group of exponent 2; moreover, because $-1 \notin T$, we have $1 \neq -1$ in $G_T(S)$.

Given an S-form $\varphi = \langle\, a_1, \ldots, a_n \,\rangle$, write $\varphi^T = \langle\, a_1^T, \ldots, a_n^T \,\rangle$ for the image of φ in $G_T(S)$.

We now generalize the construction in 8.9 of [**DM8**]. Let S be a T-subgroup of a ring A. For $a, b \in S$

$$D^v_{S,T}(a, b) \;=\; \{c \in S : \exists\, s, t \in T \text{ such that } c \;=\; sa + tb\}$$

is the set of elements of S **value-represented mod *T*** by $\langle\, a, b \,\rangle$. Since $0, 1 \in T$, it is clear that $\{x, y\} \subseteq D^v_{S,T}(x, y)$.

If $S = A^{\times}$, we are back to the case presented in 8.9 of [**DM8**], and write $D^v_T(a, b)$ for $D^v_{A^{\times}, T}(a, b)$ and $G_T(A)$ for $G_T(A^{\times})$.

Important special cases. a) If S is a ΣA^2-subgroup of A, we write a^{Σ} for a^T and $G_{red}(S)$ for $G_T(S)$, respectively. Moreover, the binary representation set $D^v_{S,\Sigma A^2}(a, b)$ will be written $D^v_{S,\Sigma}(a, b)$.

b) If S is a q-subgroup of A, we write $\widehat{s}$ for s^{A^2} and $G(S)$ for $G_{A^2}(S)$, respectively. In case $S = A^{\times}$, write $G(A) = A^{\times}/A^{\times 2}$ for $G_{A^2}(A^{\times})$. $\blacksquare$

The basic properties of binary representation sets are described in the following Lemma, whose proof is similar to those of Lemma 1.30 and Proposition 1.31 of [**DM2**], pp. 22–23.

LEMMA 2.6. *Let S be a T-subgroup of a p-ring $\langle\, A, T \,\rangle$, or a q-subgroup of A. Let x, y, u, $v \in S$ and $t \in T^{\times}$. With notation as above, we have* *a)* $uD^v_{S,T}(x,y) = D^v_{S,T}(ux,uy)$ *and* $D^v_{S,T}(x,y) = D^v_{S,T}(tx,ty)$.

b) $u \in D^v_{S,T}(x,y)$ *and* $u^T = v^T \Rightarrow v \in D^v_{S,T}(x,y)$.

c) $x^T = u^T$ *and* $y^T = v^T \Rightarrow D^v_{S,T}(x,y) = D^v_{S,T}(u,v)$.

d) $D^v_{S,T}(1,x)$ *is a subgroup of* S*. Moreover, if* T *is a preorder, then*

$$y \in D^v_{S,T}(1,x) \Rightarrow D^v_{S,T}(1,y) \subseteq D^v_{S,T}(1,x),$$

that is, $D^v_{S,T}(1,x)$ *is a saturated subgroup of* S.

e) $x \in D^v_{S,T}(1,y) \Rightarrow D^v_{S,T}(x,xy) = xD^v_{S,T}(1,y) = D^v_{S,T}(1,y)$.

f) $u \in D^v_{S,T}(x,y) \Leftrightarrow D^v_{S,T}(u,\, uxy) = D^v_{S,T}(x,y)$.

g) The following are equivalent :

(1) $(xy)^T = (uv)^T$ *and* $D^v_{S,T}(x,y) = D^v_{S,T}(u,v)$*;*

(2) $(xy)^T = (uv)^T$ *and* $D^v_{S,T}(x,y) \cap D^v_{S,T}(u,v) \neq \emptyset$.

Proof. Items (a), (b) and (c) are straightforward. For the first assertion in (d), it must be shown that $D^v_{S,T}(1,\, x)$ is closed under inverses and products. If $z \in D^v_{S,T}(1,\, x)$, then $1/z = z/z^2$, i.e., $(1/z)^T = z^T$ and (b) yields $1/z \in D^v_{S,T}(1,\, x)$. If T is a preorder, the closure of $D^v_{S,T}(1,\, x)$ under products is straightforward. If $T = A^2$, let $a,\, b \in D^v_S(1,\, x)$. Then, there are $p,\, q,\, u,\, v \in A$ such that

$$a = p^2 + q^2x \quad \text{and} \quad b = u^2 + v^2x.$$

Hence,
$$\begin{aligned} ab &= (p^2u^2 + q^2v^2x^2) + (p^2v^2 + q^2u^2)x \\ &= (p^2u^2 + q^2v^2x^2) + 2puqvx - 2puqvx + (p^2v^2 + q^2u^2)x \\ &= (pu + qvx)^2 + (pv - qu)^2x, \end{aligned}$$

and so $ab \in D^v_S(1,\, x)$. Next, assume T is a preorder, $y \in D^v_{S,T}(1,\, x)$ and $z \in D^v_{S,T}(1,\, y)$. Thus, there are $p_1,\, q_1,\, p_2,\, q_2 \in T$ such that

$$y = p_1 + q_1x \quad \text{and} \quad z = p_2 + q_2y.$$

Substituting the value of y given by the first equation into the second, yields $z = (p_2 + p_1q_2) + q_1q_2x$, whence $z \in D^v_{S,T}(1,\, x)$, as needed.

e) The first equality is a particular case of (a); for the second, since $D^v_{S,T}(1,\, y)$ is a subgroup of S and $x \in D^v_{S,T}(1,\, y)$, we have

$$xD^v_{S,T}(1,\, y) \subseteq D^v_{S,T}(1,\, y) \quad \text{and} \quad (1/x)D^v_{S,T}(1,\, y) \subseteq D^v_{S,T}(1,\, y),$$

establishing the desired equality.

f) Since $u \in D^v_{S,T}(u,\, uxy)$, only the implication ($\Rightarrow$) needs proof. From $u \in D^v_{S,T}(x,y)$ and (a) we obtain

$$ux \in xD^v_{S,T}(x,\, y) = D^v_{S,T}(x^2,\, xy) = D^v_{S,T}(1,\, xy).$$

Hence, (e) yields $D^v_{S,T}(ux,\, ux^2y) = D^v_{S,T}(ux,\, uy) = D^v_{S,T}(1,\, xy)$. Multiplying this last equality by x we get $D^v_{S,T}(u,\, uxy) = D^v_{S,T}(x,\, y)$.

g) It suffices to check that (2) $\Rightarrow$ (1); fix $z \in D^v_{S,T}(x, y) \cap D^v_{S,T}(u, v)$. Then, $(zxy)^T = (zuv)^T$, while (c) and (f) yield

$$D^v_{S,T}(x, y) = D^v_{S,T}(z, zxy) = D^v_{S,T}(z, zuv) = D^v_{S,T}(u, v),$$

ending the proof. ■

Define a binary relation, $\equiv^S_T$ on $G_T(S) \times G_T(S)$, called **binary isometry mod T**, as follows: for $a, b, c, d \in S$

$$(\equiv^S_T) \qquad \langle a^T, b^T \rangle \equiv^S_T \langle c^T, d^T \rangle \quad \Leftrightarrow \quad \begin{cases} a^T b^T = c^T d^T \text{ and} \\ D^v_{S,T}(a, b) = D^v_{S,T}(c, d). \end{cases}$$

If $S = A^\times$, write $\equiv_T$ (or $\equiv^A_T$) for $\equiv^{A^\times}_T$ and if $T = A^2$, we write $\equiv^S$ for $\equiv^S_{A^2}$.

Lemma 2.6 yields

LEMMA 2.7. *Let S be a T-subgroup of A.*

a) The structure $\langle G_T(S), \equiv^S_T, -1 \rangle$ is a proto-special group (1.9.(*a*))*. Moreover,*

(1) *For $x, y, z \in S$, $z \in D^v_{S,T}(x,y) \Leftrightarrow z^T \in D_{G_T(S)}(x^T, y^T)$.*

(2) *For $u, v, x, y \in S$,*
$$D^v_{S,T}(u,v) \subseteq D^v_{S,T}(x,y) \Leftrightarrow D_{G_T(S)}(u^T, v^T) \subseteq D_{G_T(S)}(x^T, y^T).$$

(3) *For all $x \in S$, $D_{G_T(S)}(1, x^T)$ is a subgroup of $G_T(S)$, which is saturated if T is a preorder.*

b) If binary value representation is 2**-transversal***, i.e., for all $a, b \in S$,*

$$D^v_{S,T}(a,b) = \{c \in S : \exists\, s, t \in T^\times \text{ such that } c = sa + tb\},$$

then $G_T(S)$ is a **pre-special group** (1.9.(*b*)).

c) If T is a preorder, then $G_T(S)$ is a reduced π-SG (1.9.(*a*))*. In case $T = A^2$, the following are equivalent :*

(1) *$G(S)$ is reduced;*

(2) $(A^2 + A^2) \cap S \subseteq A^2$.

In particular, if A is semi-real and Pythagorean (i.e., $\Sigma A^2 = A^2$), $G(S)$ is reduced, for all q-subgroups S of A.

PROOF. a) It is straightforward that $G_T(S)$ verifies the axioms [SG 0], [SG 1], [SG 3] and [SG 5] in 1.9.(a) and we comment only on [SG 2]. Since $2 \in A^\times$, we may write

$$u = ((1+u)/2)^2 - ((1-u)/2)^2,$$

showing that if $u \in S$, then $u \in D^v_{S,T}(1, -1)$. Since $u^T(-u^T) = 1(-1)$, the definition of $\equiv^S_T$ entails $\langle u^T, -u^T \rangle \equiv^S_T \langle 1, -1 \rangle$.

To establish ($\Rightarrow$) in item (1), for $x, y, z \in S$, 2.6.(f) yields

$$\text{(I)} \qquad z \in D^v_{S,T}(x, y) \Leftrightarrow D^v_{S,T}(z, zxy) = D^v_{S,T}(x, y).$$

Since $z^T(z^T x^T y^T) = x^T y^T$, the right-hand side of (I) and the definition of $\equiv^S_T$ yield $\langle z^T, z^T x^T y^T \rangle \equiv^S_T \langle x^T, y^T \rangle$. Now, the definition of representation in a π-SG (1.9.(g)) entails $z^T \in D_{G_T(S)}(x^T, y^T)$.

For the converse, the definition of representation in $G_T(S)$ and the discriminant axiom [SG 3] entail $\langle z^T, z^T x^T y^T \rangle \equiv_T^S \langle x^T, y^T \rangle$ (see also 1.13.(a)). Now, the definition of $\equiv_T^S$ implies $D^v_{S,T}(z,\, zxy) = D^v_{S,T}(x,\, y)$ and the equivalence in (I) yields $z \in D^v_{S,T}(x,\, y)$, as needed. Item (2) follows easily from (1), while (3) is a straightforward consequence of (1), (2) and 2.6.(d).

b) If $\langle u^T, v^T \rangle \equiv_T^S \langle x^T, y^T \rangle$, then $(uv)^T = (xy)^T$ and the equivalence (*) in 2.5 yields $(-xu)^T = (-vy)^T$. By Lemma 2.6.(g) and the definition of $\equiv_T^S$, the desired conclusion is equivalent to $u \in D^v_{S,T}(-v,\, y)$. Since $y^T \in D_{G_T(S)}(u^T,\, v^T)$, (a.1) guarantees that $y \in D^v_{S,T}(u,\, v)$. Now, because value representation is 2-transversal, there are $s,\, t \in T^\times$ such that $y = su + tv$, and so, $u = y/s + (t/s)(-v)$. Since $1/s$ and t/s are both in T, we obtain $u \in D^v_{S,T}(y,\, -v)$, as needed.

c) If T is a preorder and $\langle a^T, a^T \rangle \equiv_T^S \langle 1, 1 \rangle$, then (a.1) entails $a \in D^v_{S,T}(1,\, 1)$, that is, a is a sum of elements in T and hence itself an element of T. Since $a \in S \subseteq A^\times$, we get $a \in T^\times$ and so $a^T = 1$.

For the case of squares, if $G(S)$ is reduced and $a \in (A^2 + A^2) \cap S$, then $a = p^2 + q^2$, and so $a \in D^v_S(1,\, 1)$. By (a.1), we obtain $\widehat{a} \in D_{G(S)}(1,\, 1)$ and reducedness entails $\widehat{a} = 1$, that is, $a \in (A^2)^\times$, proving (1) $\Rightarrow$ (2). For the converse, clearly $-1 \notin (A^2)^\times$ entails $1 \neq -1$ in $G(S)$. If $\langle \widehat{a}, \widehat{a} \rangle \equiv^S \langle 1, 1 \rangle$ in $G(S)$, then $\widehat{a} \in D_{G(S)}(1, 1)$ and so (by (a.1)) $a \in D^v_S(1,\, 1)$, which immediately entails $a \in S$ to be a sum of two squares in A. The last statement follows immediately, ending the proof. ■

Note. The last assertion in 2.7.(a.3) is false if $T = A^2$, even for fields: in the special group of square classes of the rationals, $G(\mathbb{Q}) = \mathbb{Q}^\times / \mathbb{Q}^{\times 2}$, we have $1 \in D^v(1,\, 3)$, but an elementary argument shows that $2 \in D^v(1,\, 1)$ is not in $D^v(1,\, 3)$. ■

As in Definition 1.7.(d), we extend binary isometry in $G_T(S)$ to forms of arbitrary dimension. The resulting (isometry) relation will still be denoted by $\equiv_T^S$.

In fact, the π-SG associated to a p-ring is a *fully geometrical functor*, i.e., it preserves arbitrary products and right-directed colimits. To ease presentation, we recall the statement of Lemma 8.14 in [**DM8**]:

LEMMA 2.8. *A p-ring morphism, $h : \langle A_1, T_1 \rangle \longrightarrow \langle A_2, T_2 \rangle$, induces a morphism of π-SGs,*

$$(*) \qquad h^\pi : G_{T_1}(A_1) \longrightarrow G_{T_2}(A_2), \ \textit{given by} \ \ h^\pi(a^{T_1}) = h(a)^{T_2}.$$

Furthermore, $Id^\pi_{A_1} = Id_{G_{T_1}(A_1)}$ and if $g : \langle A_2, T_2 \rangle \longrightarrow \langle A_3, T_3 \rangle$ is a morphism of p-rings, then $(g \circ h)^\pi = g^\pi \circ h^\pi$. ■

We now state

PROPOSITION 2.9. *With notation as in* 2.8*, the π-SG functor from* **p-Ring** *to* **π-SG***, given by*

$$\begin{cases} \langle A, T \rangle & \longmapsto \quad G_T(A) \\ \langle A_1, T_1 \rangle \xrightarrow{h} \langle A_2, T_2 \rangle & \longmapsto \quad G_{T_1}(A_1) \xrightarrow{h^\pi} G_{T_2}(A_2) \end{cases}$$

is fully geometrical. In particular, it preserves reduced products.

PROOF. It is straightforward that the argument given in the proof of Proposition 6.4 in [**DM8**] for binary products extends to arbitrary products, while the preservation of colimits is established in full in the same result. The preservation of reduced products follows from the preservation of products, right-directed colimits and the following well-known (folklore)

FACT 2.10. *Let A_i, $i \in I$, be a family of L-structures and let D be a filter on I. For $U \subseteq V \subseteq I$, let*

$$A(U) \ = \ \prod_{i \in U} A_i \quad \text{and} \quad p_{VU} : A(V) \longrightarrow A(U),$$

be the projection that forgets the coordinates outside U. Then, the system $\mathcal{A} = \{A(U); \{p_{VU} : V \supset U \in D\}\}$ is an inductive system over the right-directed poset $\langle D, \subseteq^{op} \rangle$, and its colimit is naturally isomorphic to the reduced product $\prod_D A_i$. ■

3. The K-theory of a Ring and of its Proto-Special Group

The main result of this section is Theorem 2.16, a broad generalization of Theorem 4.12 in [**DM8**].

The K-theory of special groups was introduced in [**DM3**] and further developed in [**DM7**]. In order to establish 2.16, we shall have to extend the definition of the K-theory of special groups set down in [**DM3**] to proto-special groups. For the reader's convenience we also succinctly recall the construction of the Milnor mod 2 K-theory of a ring, essentially due to D. Guin [**Gu**] (see [**DM8**]).

2.11. **The K-theory of a π-SG.** We follow the presentation in [**DM3**] and [**DM7**], indicating the (slight) necessary modifications. Let G be a π-SG, written multiplicatively. Let k_1G be G written additively, that is, we **fix** an isomorphism

$$\lambda : G \longrightarrow k_1G, \quad \text{such that } \lambda(ab) = \lambda(a) + \lambda(b), \text{ for all } a, b \in G.$$

Note that $\lambda(1)$ is the zero in k_1G and that k_1G has exponent 2, i.e., $\lambda(a) = -\lambda(a)$, for $a \in G$. Define the K-theory of G as the $\mathbb{F}_2$-algebra

$$k_*G = (\mathbb{F}_2, k_1G, \dots, k_nG, \dots)$$

obtained as the quotient of the graded tensor algebra over $\mathbb{F}_2$

$$(\mathbb{F}_2, k_1G, \dots, \underbrace{k_1G \otimes \dots \otimes k_1G}_{n \text{ times}}, \dots),$$

by the ideal generated by $\{\lambda(a) \otimes \lambda(ab) : a \in D_G(1, b)\}$. Thus, for each $n \geq 2$, k_nG is the quotient of the n-fold tensor product $k_1G \otimes \dots \otimes k_1G$ over $\mathbb{F}_2$, by the subgroup consisting of finite sums of elements of the form $\lambda(a_1) \otimes \lambda(a_2) \otimes \dots \otimes \lambda(a_n)$, where for some $1 \leq i \leq n-1$ and some $b \in G$ we have $a_{i+1} = a_ib$ and $a_i \in D_G(1, b)$. As usual, we write $\lambda(a_1)\lambda(a_2)\dots\lambda(a_n)$ for $\lambda(a_1) \otimes \lambda(a_2) \otimes \dots \otimes \lambda(a_n)$.

It is straightforward that for all $n \geq 1$ and $\eta \in k_nG$ we have $\eta + \eta = 0$, that is, k_nG is a group of exponent 2. Note that for all $a, b \in G$, $\lambda(a)\lambda(ab) = \lambda(a)^2 + \lambda(a)\lambda(b)$. Thus,

$$(*) \qquad a \in D_G(1, b) \quad \text{implies} \quad \text{In } k_2G, \ \lambda(a)\lambda(ab) = 0$$

or equivalently, $\lambda(a)^2 = \lambda(a)\lambda(b)$. ■

The statement and proof of Proposition 2.1 in [**DM3**] hold *verbatim* for π-SGs:

Proposition 2.12. *Let G be a π-SG and let a, b, $a_1, \ldots, a_n$ be elements in G. Let σ be a permutation of $\{1, \ldots, n\}$.*

a) In k_2G, $\lambda(a)\lambda(-a) = 0$.

b) In k_2G, $\lambda(a)^2 = \lambda(a)\lambda(-1)$.

c) In k_nG, $\lambda(a_1)\lambda(a_2)\ldots\lambda(a_n) = \lambda(a_{\sigma(1)})\lambda(a_{\sigma(2)})\ldots\lambda(a_{\sigma(n)})$, i.e., k_*G is a commutative graded ring.

Proof. For (a), since G is a π-SG, it satisfies axiom [SG 2] in Definition 1.9.(a) and so $a \in D_G(1, -1)$, for every $a \in G$; the conclusion is now immediate from (*) in 2.11. The remaining items can be obtained with the same proof as that of Proposition 2.1 in [**DM3**]. ■

2.13. **The mod 2 *K*-theory of a Ring.** Here we recall the mod 2 K-theory of a ring introduced in [**DM8**], patterned after the construction in section 3 of [**Gu**]. Let A be a ring. We set $K_0A = \mathbb{Z}$ and let K_1A be $A^\times$ written additively, that is, we fix an isomorphism

$$l : A^\times \longrightarrow K_1A, \text{ such that } l(ab) = l(a) + l(b), \ \forall\, a, b \in A^\times.$$

Then, Milnor's K-theory of A is the graded ring (Def. 3.2, p. 47, [**Gu**])

$$K_*A = \langle \mathbb{Z}, K_1A, \ldots, K_nA, \ldots \rangle,$$

obtained as the quotient of the graded tensor algebra over $\mathbb{Z}$,

$$\langle \mathbb{Z}, K_1A, \ldots, \underbrace{K_1A \otimes \ldots \otimes K_1A}_{n \text{ times}}, \ldots \rangle$$

by the ideal generated by

$$\{l(a) \otimes l(b) : a, b \in A^\times \text{ and } a + b = 1 \text{ or } 0\}.$$

Hence, for each $n \geq 2$, K_nA is the quotient of the n-fold tensor product over $\mathbb{Z}$, $K_1A \otimes \ldots \otimes K_1A$, by the subgroup consisting of sums of generators $l(a_1) \otimes \ldots \otimes l(a_n)$, such that for some $1 \leq i \leq n-1$, $a_i + a_{i+1} = 1$ or 0. As usual, we shall write the generators in K_nA as $l(a_1) \cdots l(a_n)$, omitting the tensor symbol.

We define the **mod 2 *K*-theory** of A, as the graded ring

$$k_*A = \langle k_0A, k_1A, \ldots, k_nA, \ldots \rangle =_{def} K_*A/2K_*A,$$

that is, for each $n \geq 0$, k_nA is the quotient of K_nA by the subgroup $\{2\eta \in K_nA : \eta \in K_nA\}$. We have $k_0A = \mathbb{F}_2$ and $k_1A \approx A^\times/(A^\times)^2$, via an isomorphism still denoted by l. A generator in k_nA will be written $l(a_1) \cdots l(a_n)$. Clearly, k_nA is a group of exponent 2, i.e., $\eta + \eta = 0$, for all $\eta \in k_nA$. ■

Lemma 2.14. *If A is a ring, b, a, $a_1, \ldots, a_n \in A^\times$ and σ is a permutation of $\{1, \ldots, n\}$, then*

a) In k_2A, $l(a)l(-a) = 0$.

b) In k_2A, $l(a)l(-1) = l(a)^2$.

c) In k_2A, $l(a)l(b) = l(b)l(a)$.

d) In k_nA, $l(a_1) \cdots l(a_n) = l(a_{\sigma(1)}) \cdots l(a_{\sigma(n)})$.

e) If $t_1, \ldots, t_n \in A^\times$, then in k_nA, $l(t_1^2a_1) \cdots l(t_n^2a_n) = l(a_1) \cdots l(a_n)$.

PROOF. a) Since $a + (-a) = 0$, we get $l(a)l(-a) = 0$ in K_2A, and so the same is true in k_2A. The remaining items follow from (a) and the fact that k_nA is a group of exponent two ($n \geq 0$), with the same proof as in Lemma 4.10 in [**DM8**]. ■

LEMMA 2.15. *Let A be a ring satisfying the transversality assumption in* 2.7.(*b*) *for $T = A^2$, i.e., for all $a, b \in A^\times$ (cf.* 2.5)

$$D^v(a,b) := D^v_{A^\times, A^2}(a,b) = \{c \in A^\times : \exists\, x, y \in A^\times \text{ such that } c = x^2a + y^2b\}.$$

Let $a, b, a_1, \ldots, a_n \in A^\times$, with $a \in D^v(1,b)$. If $a_i = a$ and $a_j = ab$ for some $1 \leq i \neq j \leq n$, then $l(a_1)\cdots l(a_n) = 0$ in k_nA.

PROOF. Since A verifies the transversality assumption with respect to squares and $a \in D^v(1, b)$, there are $p, q \in A^\times$ such that $a = p^2 + q^2b$. Hence,

$$1 = (p^2/a^2)\, a + (q^2/a^2)\, ba = (p/a)^2a + (q/a)^2ab,$$

and so, the definition of k_*A and 2.14.(e) yield $l(a)l(ab) = 0$ in k_2A. The general statement follows immediately from 2.14.(d). ■

THEOREM 2.16. *If A is a ring satisfying the transversality assumption in* 2.7.(*b*) *for $T = A^2$, then the rules $\alpha_0 = Id_{\mathbb{F}_2}$ and, for $n \geq 1$,*

$$\alpha_n : k_nA \longrightarrow k_nG(A) \textit{ is defined on generators by}$$

$$\alpha_n(l(a_1)\ldots l(a_n)) = \lambda(\widehat{a_1})\ldots\lambda(\widehat{a_n}),$$

determine a graded ring isomorphism, $\alpha = \{\alpha_n : n \geq 0\}$, between Milnor's mod 2 *K-theory of A and the K-theory of the pre-special group $G(A)$.*

PROOF. Recall that $G(A) = A^\times/(A^2)^\times = \{\widehat{a} : a \in A^\times\}$ is, by Lemma 2.7.(b), the *pre-special group* associated to A. To simplify notation, write G for $G(A)$. Define $f : K_1A \longrightarrow k_1G$ by $f(l(a)) = \lambda(\widehat{a})$; clearly f is a surjective group homomorphism. Notice that for $a, b \in A^\times$, $\widehat{a} = \widehat{b}$ iff $f(l(a)) = f(l(b))$. By the definition of k_1A, f factors through an isomorphism

$$\alpha_1 : k_1A \longrightarrow k_1G, \quad l(a) \mapsto \lambda(\widehat{a}).$$

For $n \geq 2$, define $f_n : \underbrace{K_1A \times \ldots \times K_1A}_{n \text{ times}} \longrightarrow k_nG$, by

$$f_n((l(a_1), \ldots, l(a_n)) = \lambda(\widehat{a_1})\cdots\lambda(\widehat{a_n}).$$

It is straightforward to show that f_n is n-linear. Therefore, f_n induces a homomorphism γ_n from the n-fold tensor product $\bigotimes_{i=1}^n K_1A$ to k_nG, taking $l(a_1)\ldots l(a_n)$ to $\lambda(\widehat{a_1})\cdots\lambda(\widehat{a_n})$. To show that γ_n factors through K_nA it is enough to verify that for $a, b \in A^\times$,

$$(*) \qquad a + b = 0 \ \text{ or } \ a + b = 1 \quad \Rightarrow \quad \lambda(\widehat{a})\lambda(\widehat{b}) = 0 \ \text{ in } k_2G.$$

If $b = -a$, the conclusion in (*) follows from 2.12.(a); if $b = 1 - a$, then since $\widehat{-a} = -\widehat{a}$ and $b \in D^v(1, -a)$, we get $\widehat{b} \in D_G(1, -\widehat{a})$. Thus, taking into account 2.12.(b), we obtain

$$\begin{aligned} 0 &= \lambda(\widehat{b})\lambda(-\widehat{a}\widehat{b}) = \lambda(\widehat{b})[\lambda(-1) + \lambda(\widehat{a}) + \lambda(\widehat{b})] \\ &= \lambda(-1)\lambda(\widehat{b}) + \lambda(\widehat{b})\lambda(\widehat{a}) + \lambda(\widehat{b})^2 = \lambda(\widehat{b})\lambda(\widehat{a}), \end{aligned}$$

as needed. Hence, γ_n factors through K_nA to yield a group homomorphism,

$$\mu_n : K_nA \longrightarrow k_nG,$$

taking $l(a_1)\ldots l(a_n)$ to $\lambda(\widehat{a_1})\cdots\lambda(\widehat{a_n})$. If some $a_i \in (A^2)^\times$, then $\widehat{a_i} = 1$ in G and so $\lambda(\widehat{a_i}) = 0$ in k_1G, which in turn forces $\lambda(\widehat{a_1})\cdots\lambda(\widehat{a_n}) = 0$ in k_nG. This shows that μ_n factors uniquely through k_nA, to yield a homomorphism $\alpha_n : k_nA \longrightarrow k_nG$, mapping a generator $l(a_1)\cdots l(a_n)$ in k_nA to the generator $\lambda(\widehat{a_1})\cdots\lambda(\widehat{a_n})$ in k_nG. Clearly, if $\xi \in k_nA$ and $\eta \in k_mA$, then $\alpha_{n+m}(\xi\ \eta) = \alpha_n(\xi)\alpha_m(\eta)$, and so $\alpha = \{\alpha_n : n \geq 0\}$ is a graded ring homomorphism from k_*A to k_*G.

To verify that α is an isomorphism, we shall construct a graded ring homomorphism $\theta = \{\theta_n : n \geq 0\}$ from k_*G to k_*A, such that $\alpha \circ \theta$ and $\theta \circ \alpha$ are the identities in k_*G and k_*A, respectively. θ_0 is the identity on $\mathbb{F}_2$ and θ_1 is the inverse of α_1.

For $n \geq 2$, consider the map $h_n : (k_1G)^n \longrightarrow k_nA$, defined by

$$(\lambda(\widehat{a_1}), \ldots, \lambda(\widehat{a_n})) \longmapsto l(a_1)\ldots l(a_n).$$

To show that h_n is well-defined, if $\lambda(\widehat{a_i}) = \lambda(\widehat{b_i})$, $1 \leq i \leq n$, then $\widehat{a_i} = \widehat{b_i}$; thus, for each $1 \leq i \leq n$, there is $t_i \in A^\times$ such that $t_i^2 b_i = a_i$. Consequently, Lemma 2.14.(e) yields

$$l(a_1)\ldots l(a_n) = l(t_1^2 b_1)\ldots l(t_n^2 b_n) = l(b_1)\ldots l(b_n).$$

It is straightforward to check that h_n is n-linear. Thus, there is a homomorphism δ_n from $\bigotimes_{i=1}^n k_1G$ to k_nA, mapping $\lambda(\widehat{a_1})\ldots\lambda(\widehat{a_n})$ to $l(a_1)\ldots l(a_n)$. For δ_n to factor through k_nG, it is enough to verify

$$\text{(I)} \qquad \widehat{a} \in D_G(1, \widehat{b}) \quad \Rightarrow \quad l(a)l(ab) = 0 \ \text{ in } k_2A.$$

The antecedent in (I) and 2.7.(a.1) imply $a \in D^v(1, b)$ in A and Lemma 2.15 yields $l(a)l(ab) = 0$, as needed. Hence, δ_n induces a homomorphism, $\theta_n : k_nG \longrightarrow k_nA$, such that for all $a_1, \ldots, a_n \in A^\times$, $\theta_n(\lambda(\widehat{a_1})\ldots\lambda(\widehat{a_n})) = l(a_1)\ldots l(a_n)$. Clearly, $\theta = \{\theta_n : n \geq 0\}$ is a graded ring homomorphism. By their definition on generators, $\alpha_n \circ \theta_n$ is the identity on k_nG, while $\theta_n \circ \alpha_n$ is the identity on k_nA, ending the proof. ∎

4. T-isometry

In this section we introduce a notion of isometry of S-forms modulo a preorder T that will be fundamental in all that follows; this notion was suggested by Proposition 5.5.2 (p.43) of [**Wa**].

Definition 2.17. *Let $\langle A, T\rangle$ be a p-ring or $T = A^2$, and let S be a T-subgroup of A. Let φ, ψ be S-forms of dimension $n \geq 1$. Recall that $\approx$ denotes matrix isometry of forms, cf.* 2.4(b). *We say that φ is **T-isometric** to ψ, written $\boldsymbol{\varphi \approx^S_T \psi}$, if there is a sequence, $\varphi_0, \varphi_1, \ldots, \varphi_k$, of n-dimensional forms over S, such that*

(*i*) $\varphi_0 = \varphi$ *and* $\varphi_k = \psi$;

(*ii*) *For all $1 \leq i \leq k$, either $\varphi_i \approx \varphi_{i-1}$, or $\varphi_i = \langle t_1x_1, \ldots, t_nx_n\rangle$, with $t_1, \ldots, t_n \in T^\times$ and $\varphi_{i-1} = \langle x_1, \ldots, x_n\rangle$.*

REMARK 2.18. a) If $T = A^2$ in Definition 2.17, for all S-forms φ, ψ, of the same dimension we have,

$$\varphi \approx^S_{A^2} \psi \quad \Leftrightarrow \quad \varphi \approx \psi.$$

This follows easily from the definition of $\approx^S_{A^2}$ and 2.4(c.1). Hence, our notion of T-isometry generalizes (and reduces to) matrix isometry in the case of quadratic form theory modulo squares.

b) If $\langle A, T \rangle$ is a p-ring and $S = A^\times$, write $\approx_T$ (or $\approx^A_T$) for $\approx^S_T$. ∎

LEMMA 2.19. *Let $\langle A, T \rangle$ be a p-ring, or $T = A^2$. Let S be a T-subgroup of A and let φ, ψ be forms of the same dimension over S.*

a) For each $n \geq 1$, $\approx^S_T$ is an equivalence relation on the set of n-forms over S.

b) (1) $\varphi \approx \psi \quad \Rightarrow \quad \varphi \approx^S_T \psi$;

(2) *If $\varphi = \langle t_1a_1, \ldots, t_na_n \rangle$, where $\{t_1, \ldots, t_n\} \subseteq T^\times$, and $\psi = \langle a_1, \ldots, a_n \rangle$, then $\varphi \approx^S_T \psi$.*

c) For all $a \in S$, $\varphi \approx^S_T \psi \;\Rightarrow\; a\varphi \approx^S_T a\psi$.

d) If θ is an S-form, then

(1) $\varphi \approx^S_T \psi \;\Rightarrow\; \theta \oplus \varphi \;\approx^S_T\; \theta \oplus \psi$;

(2) $\varphi \approx^S_T \psi \;\Rightarrow\; \theta \otimes \varphi \;\approx^S_T\; \theta \otimes \psi$.

(3) *The operations $\oplus$ and $\otimes$ are associative with respect to $\approx^S_T$.*

e) If φ_i, ψ_i, are S-forms of the same dimension, $i = 1, 2$, then

$$\varphi_1 \approx^S_T \psi_1 \quad and \quad \varphi_2 \approx^S_T \psi_2 \quad \Rightarrow \quad \begin{cases} \varphi_1 \oplus \varphi_2 & \approx^S_T \quad \psi_1 \oplus \psi_2; \\ \varphi_1 \otimes \varphi_2 & \approx^S_T \quad \psi_1 \otimes \psi_2. \end{cases}$$

f) $\varphi \approx^S_T \psi \;\Rightarrow\; d(\varphi)^T = d(\psi)^T$, where $d(\varphi)$ is the discriminant of φ.

g) If $\dim \varphi = \dim \psi = n \geq 1$ and σ is a permutation of $\{1, \ldots, n\}$, then

$$\varphi \approx^S_T \psi \;\Rightarrow\; \varphi^\sigma \approx^S_T \psi^\sigma,$$

where $\varphi^\sigma = \langle a_{\sigma(1)}, \ldots, a_{\sigma(n)} \rangle$. Hence, the operations $\oplus$ and $\otimes$ are commutative with respect to $\approx^S_T$.

h) For $a, b \in S$, $\langle a \rangle \approx^S_T \langle b \rangle \;\Leftrightarrow\; a^T = b^T$.

PROOF. a) Clearly, $\approx^S_T$ is reflexive. Let $\varphi_0, \varphi_1, \ldots, \varphi_k$ be a sequence witnessing $\varphi \approx^S_T \psi$. Then, the sequence $\varphi_k, \varphi_{k-1}, \ldots, \varphi_0$ witnesses $\psi \approx^S_T \varphi$. If θ is a S-form such that $\varphi \approx^S_T \psi \approx^S_T \theta$, let $\varphi_0, \varphi_1, \ldots, \varphi_k$ and $\psi_0, \psi_1, \ldots, \psi_m$ be sequences of S-forms witnessing the first and second of these T-isometries. It is straightforward that $\varphi_0, \varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_m$ is a sequence guaranteeing that $\varphi \approx^S_T \psi$.

b) This is clear.

c) Recall that $\varphi \approx \psi$ entails $a\varphi \approx a\psi$. Indeed, if $M \in \mathrm{GL}_n(A)$, where $n = \dim \varphi = \dim \psi$, is such that $M\mathscr{M}(\varphi)M^t = \mathscr{M}(\psi)$, it is obvious that $M\mathscr{M}(a\varphi)M^t = \mathscr{M}(a\psi)$. With this in hand, if $\varphi_0, \varphi_1, \ldots, \varphi_k$ is a sequence of S-forms witnessing $\varphi \approx^S_T \psi$, then the sequence $a\varphi_i$, $1 \leq i \leq k$, witnesses $a\varphi \approx^S_T a\psi$.

d) (1) Let $\varphi_0, \varphi_1, \ldots, \varphi_k$ be a sequence witnessing $\varphi \approx_T^S \psi$. We claim that $\theta_i = \theta \oplus \varphi_i$, $1 \le i \le k$, testifies to the desired conclusion. Indeed, for $0 \le i \le k$, we have two possibilities (corresponding to the two alternatives in 2.17 (ii)):

∗ If $\varphi_i \approx \varphi_{i-1}$, then, since matrix isometry preserves orthogonal sums, we have

$$\theta_i = \theta \oplus \varphi_i \approx \theta \oplus \varphi_{i-1} = \theta_{i-1};$$

∗ If $\varphi_i = \langle t_1a_1, \ldots, t_na_n \rangle$, with $\varphi_{i-1} = \langle a_1, \ldots, a_n \rangle$ and $\{t_1, \ldots, t_n\} \subseteq T^\times$, then, since $1 \in T^\times$, it is clear that the entries $\theta \oplus \varphi_i$ are multiples in $T^\times$ of the entries in $\theta \oplus \varphi_{i-1}$, establishing our claim. Item (2) is a straightforward consequence of (c) and (1) (see the proof of 1.13.(c)), while (3) is clear.

e) By (d.1), adding φ_1 to both sides of $\varphi_2 \approx_T^S \psi_2$ yields $\varphi_1 \oplus \varphi_2 \approx_T^S \varphi_1 \oplus \psi_2$. Now, adding ψ_2 to both sides of $\varphi_1 \approx_T^S \varphi_2$ gives $\varphi_1 \oplus \psi_2 \approx_T^S \varphi_2 \oplus \psi_2$, and the transitivity of $\approx_T^S$ yields the desired conclusion. Similarly, one proves that $\approx_T^S$ preserves tensor products.

f) Recall that (cf. 2.4(a.3))

(I) $$\varphi \approx \psi \quad \Rightarrow \quad d(\varphi)d(\psi) \in {A^\times}^2 \subseteq T^\times.$$

Let $\varphi_0, \varphi_1, \ldots, \varphi_k$ be a sequence of S-forms witnessing $\varphi \approx_T^S \psi$. By induction on $0 \le i \le k$, we show $d(\varphi)d(\varphi_i) \in T^\times$, this being obvious for $i = 0$. Assume it true for $i < k$; again, there are two possibilities:

∗ If $\varphi_i \approx \varphi_{i+1}$, the conclusion follows from (I);

∗ If $\varphi_{i+1} = \langle t_1a_1, \ldots, t_na_n \rangle$, with $\varphi_i = \langle a_1, \ldots, a_n \rangle$, then

$$d(\varphi_{i+1}) = t_1t_2 \cdot \ldots \cdot t_n \cdot d(\varphi_i)$$

and the induction hypothesis yields $d(\varphi)d(\varphi_{i+1}) = t_1t_2 \cdot \ldots \cdot t_n \cdot d(\varphi)d(\varphi_i) \in T^\times$, completing the induction step.

g) Since $\approx_T^S$ is transitive (by (a)), it is enough to show that $\varphi \approx_T^S \varphi^\sigma$. But this follows immediately from the well-known fact that $\theta \approx \theta^\sigma$. It is then clear that $\oplus$ is commutative with respect to $\approx_T^S$, whence the same is true for $\otimes$.

h) Implication ($\Rightarrow$) follows from (f) and $a^T = d(\langle a \rangle)^T$. For the converse, if $s \in T^\times$ is such that $b = sa$, the sequence $\langle a \rangle$, $\langle sa \rangle$ ($= \langle b \rangle$) entails $\langle a \rangle \approx_T^S \langle b \rangle$. ■

Definition 2.20. *Let A be a ring and let T be a proper preorder of A or $T = A^2$. A form φ of dimension ≥ 2 over a T-subgroup S of A is:*

a) ***T*-isotropic** *if there is a form ψ so that $\varphi \approx_T^S \langle 1, -1 \rangle \oplus \psi$.*

b) ***T*-hyperbolic** *if $\varphi \approx_T^S k\langle 1, -1 \rangle$ for some integer $k \ge 1$.* ■

Lemma 2.21. *a) Let $\langle A, T \rangle$ be a p-ring and let S be a T-subgroup of A. If binary value representation satisfies the transversality condition in 2.7.(b), then for all a, b, c, $d \in S$, the following are equivalent:*

(1) $\langle a^T, b^T \rangle \equiv_T^S \langle c^T, d^T \rangle$;

(2) *There are u, v, x, $y \in T^\times$ and $M \in \mathrm{GL}_2(A)$ such that*

$$M \begin{pmatrix} ua & 0 \\ 0 & vb \end{pmatrix} M^t = \begin{pmatrix} xc & 0 \\ 0 & yd \end{pmatrix},$$

that is, $\langle ua, vb \rangle \approx \langle xc, yd \rangle$;

(3) $\langle a, b \rangle \approx_T^S \langle c, d \rangle$.

In case $T = A^2$, we can dispense with the transversality assumption and have:
b) For all $u, v, x, y \in S$, $\langle \widehat{u}, \widehat{v} \rangle \equiv^S \langle \widehat{x}, \widehat{y} \rangle \Leftrightarrow \langle u, v \rangle \approx \langle x, y \rangle$.

PROOF. a) We show that (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): By definition of $\equiv^S_T$, $a^T b^T = c^T d^T$ and $D^v_{S,T}(a, b) = D^v_{S,T}(c, d)$. These equations, together with the transversality assumption yield $t, t_1, t_2 \in T^\times$ such that $d = abct$ and $c = t_1 a + t_2 b$. Hence, $d = abct = tab(t_1 a + t_2 b) = (tt_2 b^2)a + (tt_1 a^2)b$.

Set $M = \begin{pmatrix} t_1 & t_2 \\ -bt & at \end{pmatrix}$; then $det(M) = tt_1 a + tt_2 b = t(t_1 a + t_2 b) = tc \in S$, and so $M \in \mathrm{GL}_2(A)$. Straightforward computation gives,

$$M \begin{pmatrix} t_2 a & 0 \\ 0 & t_1 b \end{pmatrix} M^t = \begin{pmatrix} (t_1 t_2)c & 0 \\ 0 & td \end{pmatrix},$$

and so it suffices to take $u = t_2$, $v = t_1$, $x = t_1 t_2$ and $y = t$ (all units in T) to obtain the desired conclusion.

(2) $\Rightarrow$ (1) : Assume that there is $M \in \mathrm{GL}_2(A)$ verifying (2); then $xycd = det(M)^2 uvab$ and so $a^T b^T = c^T d^T$. Clearly, $xc = s^2 ua + t^2 vb$, where (s, t) is the first row of M. Since $x, u, v \in T^\times$, we conclude that $c \in D^v_{S,T}(a, b) \cap D^v_{S,T}(c, d)$ and 2.6.(g) entails $\langle a^T, b^T \rangle \equiv^S_T \langle c^T, d^T \rangle$.

(2) $\Rightarrow$ (3) : With notation as in (2), the sequence $\langle a, b \rangle$, $\langle ua, vb \rangle$, $\langle xc, yd \rangle$, $\langle c, d \rangle$ is a witness to $\langle a, b \rangle \approx^S_T \langle c, d \rangle$.

(3) $\Rightarrow$ (1) : Let $\langle a, b \rangle = \varphi_0, \varphi_1, \ldots, \varphi_k = \langle c, d \rangle$ be a sequence of binary S-forms witnessing $\langle a, b \rangle \approx^S_T \langle c, d \rangle$. By induction on $0 \le i \le k$, we show that $\langle a^T, b^T \rangle \equiv^S_T \varphi^T_i$, a fact obviously true for $i = 0$. Assume it verified for $i < k$; we have two possibilities:

(i) If $\varphi_i \approx \varphi_{i+1}$, it follows immediately from the equivalence between (1) and (2) together with 2.19.(b.1), that $\varphi^T_i \equiv^S_T \varphi^T_{i+1}$. Since binary isometry is transitive (cf. [SG 0] in 2.7.(a)), the induction hypothesis entails $\langle a^T, b^T \rangle \equiv^S_T \varphi^T_{i+1}$;

(ii) If $\varphi_{i+1} = \langle s_1 z_1, s_2 z_2 \rangle$, with $s_i \in T^\times$ and $\varphi_i = \langle z_1, z_2 \rangle$, then $\varphi^T_i = \varphi^T_{i+1}$ and the induction hypothesis yields $\langle a^T, b^T \rangle \equiv^S_T \varphi^T_{i+1}$, completing the induction step and the proof of (a).

b) ($\Rightarrow$) By the definition of $\equiv^S$,

(I) $$\widehat{uv} = \widehat{xy} \quad \text{and} \quad D^S(u, v) = D^S(x, y).$$

The equations in (I) together with the definition of $\equiv^S$ yield $e \in A^\times$ and $s, t \in A$ such that

(II) $$uxy = v/e^2 \quad \text{and} \quad u = s^2 x + t^2 y.$$

Hence,

(III) $$v = uxye^2 = x(y^2 t^2 e^2) + y(x^2 s^2 e^2).$$

Set $M = \begin{pmatrix} s & t \\ -yte & xse \end{pmatrix}$; then $det(M) = xs^2 e + yt^2 e = ue \in A^\times$, that is, $M \in \mathrm{GL}_2(A)$. It is straightforward, using the equations in (II) and (III), to show

(IV) $$M\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} M^t = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

($\Leftarrow$) If there is $M \in \mathrm{GL}_2(A)$ such that (IV) holds, then $uv = det(M)^2 xy$, and so $\widehat{uv} = \widehat{xy}$. It is clear that $u = s^2x + t^2y$, where (s, t) is the first row of M. Hence, $\widehat{uv} = \widehat{xy}$ and $u \in D^S(x, y) \cap D^S(u, v)$, and Lemma 2.6.(g) entails $\langle \widehat{u}, \widehat{v} \rangle \equiv^S \langle \widehat{x}, \widehat{y} \rangle$, ending the proof. ■

Recall (2.5) that if $\varphi = \langle a_1, \ldots, a_n \rangle$ is a form over a T-subgroup S of A, $\varphi^T = \langle a_1^T, \ldots, a_n^T \rangle$ is the image form in $G_T(S)$.

COROLLARY 2.22. *Let A be a ring and let T be A^2 or a preorder of A. If S is a T-subgroup of A and φ, ψ are S-forms of the same dimension, then $\varphi^T \equiv_T^S \psi^T \Rightarrow \varphi \approx_T^S \psi$.*

PROOF. We use induction on the common dimension, n, of φ, ψ; the case $n = 2$ is (1) $\Rightarrow$ (3) in 2.21(a). Assume the result true for forms of dimension $n \geq 2$ and let $\varphi = \langle a \rangle \oplus \theta_1$ and $\psi = \langle b \rangle \oplus \theta_2$, with dim $\theta_i = n$. If $\varphi^T \equiv_T^S \psi^T$, then there are $x, y \in S$ and a $(n-1)$-dimensional S-form, θ, such that

(1) $\langle a^T, x^T \rangle \equiv_T^S \langle b^T, y^T \rangle$; (2) $\theta_1^T \equiv_T^S \langle x^T \rangle \oplus \theta^T$;

(3) $\theta_2^T \equiv_T^S \langle y^T \rangle \oplus \theta^T$.

Lemma 2.21(a), the induction hypothesis and relations (1) $-$ (3) yield

(4) $\langle a, x \rangle \approx_T^S \langle b, y \rangle$; (5) $\theta_1 \approx_T^S \langle x \rangle \oplus \theta$; (6) $\theta_2 \approx_T^S \langle y \rangle \oplus \theta$.

Now, from Lemma 2.19.(e) and (4) $-$ (6) we obtain, recalling the associativity of $\oplus$ (2.19.(d.3)),

$$\begin{aligned}\varphi &= \langle a \rangle \oplus \theta_1 \approx_T^S \langle a, x \rangle \oplus \theta \approx_T^S \langle b, y \rangle \oplus \theta \\ &\approx_T^S \langle b \rangle \oplus (\langle y \rangle \oplus \theta) \approx_T^S \langle b \rangle \oplus \theta_2 = \psi.\end{aligned}$$ ■

2.23. **Notation.** Let A be a set and $n \geq 2$, $1 \leq k \leq n$ be integers. If $a = \langle a_1, \ldots, a_n \rangle \in A^n$, write $\langle a_1, \ldots, \check{a}_k, \ldots, a_n \rangle$ for the element of A^{n-1} obtained by omitting the k^{th}-coordinate of a. ■

In the general setting of proto-special groups associated to T-subgroups of rings, there are several notions of *representation* that are of crucial importance to distinguish. In the field case all these notions coincide.

DEFINITION 2.24. *Let T be A^2 or a preorder of A and let S be a T-subgroup of A. Let $\varphi = \langle b_1, \ldots, b_n \rangle$ be a S-form and let $\varphi^T = \langle b_1^T, \ldots, b_n^T \rangle$ be the corresponding n-form in $G_T(S)$.*

a) $D_{S,T}(\varphi) = \{a \in S : \exists\, a_2, \ldots, a_n \in S$ *such that* $\varphi^T \equiv_T^S \langle a^T, a_2^T, \ldots, a_n^T \rangle\}$

is the set of elements of S **isometry-represented mod T** *by φ* **in $G_T(S)$**.

b) $D_{S,T}^v(\varphi) = \{a \in S :$ *There are* $x_1, \ldots, x_n \in T$ *such that* $a = \sum_{i=1}^n x_i b_i\}$

is the set of elements of S **value-represented mod T** *by φ.*

c) $D_{S,T}^t(\varphi) = \{a \in S :$ *There are* $z_1, \ldots, z_n \in T^\times$ *such that* $a = \sum_{i=1}^n z_i b_i\}$

is the set of elements of S **transversally represented mod T** *by φ. Clearly, $D_{S,T}^t(\varphi) \subseteq D_{S,T}^v(\varphi)$.*

d) The set $\mathfrak{D}_{S,T}(\varphi)$ *of elements of* S **inductively represented mod T** *by* φ*, is defined as follows:*

- *If* $n = 2$, $\mathfrak{D}_{S,T}(\varphi) = D^v_{S,T}(b_1, b_2)$.
- *If* $n \geq 3$, $\mathfrak{D}_{S,T}(\varphi) = \bigcap_{k=1}^n \bigcup \{D^v_{S,T}(b_k, u) : u \in D^v_{S,T}(b_1, \ldots, \check{b}_k, \ldots, b_n)\}$.

If $S = A^\times$*, write* D_T*,* D^v_T*,* D^t_T *and* $\mathfrak{D}_T$ *for the representation sets defined above; if in addition* $T = A^2$*, the corresponding representation sets will simply be written* D*,* D^v*,* D^t *and* $\mathfrak{D}$*.* ■

Remark 2.25. All the representation sets defined in 2.24 are *subsets of* S *invariant modulo units in* T. Hence, their images modulo T are subsets of $G_T(S)$. Furthermore, the definition of representation in a π-SG (1.9.(g)) yields, for $a \in S$ and a S-form φ,

$$\text{(I)} \qquad a \in D_{S,T}(\varphi) \quad \Leftrightarrow \quad a^T \in D_{G_T(S)}(\varphi^T).$$

Since this equivalence is frequently used below, to lighten notation we shall often write "$a^T \in D(\varphi^T)$ in $G_T(S)$" for its right-hand side.

Note that for binary forms, (I) and 2.7(a.1) entail, for all s, $t \in S$, $D_{S,T}(s, t) = D^v_{S,T}(s, t)$, that is, for binary forms, value representation and isometry representation coincide. ■

Lemma 2.26. *Let* A *be a ring and let* T *be a preorder of* A*, or* $T = A^2$*. Let* S *be a* T*-subgroup of* A *and let* $n \geq 2$ *be an integer. Let* $\varphi = \langle b_1, \ldots, b_n \rangle$ *be an* S*-form and let* σ *be a permutation of* $\{1, \ldots, n\}$*. As in 2.19.(g), let* $\varphi^\sigma = \langle b_{\sigma(1)}, \ldots, b_{\sigma(n)} \rangle$*. Then,*

a) (i) $D^v_{S,T}(\varphi) = D^v_{S,T}(\varphi^\sigma)$; *(ii)* $D^t_{S,T}(\varphi) = D^t_{S,T}(\varphi^\sigma)$;

(iii) $\mathfrak{D}_{S,T}(\varphi) = \mathfrak{D}_{S,T}(\varphi^\sigma)$;

(iv) $G_T(S)$ *is a RSG* $\Rightarrow$ $D_{S,T}(\varphi) = D_{S,T}(\varphi^\sigma)$.

b) $D_{S,T}(\varphi) \subseteq D^v_{S,T}(\varphi)$.

c) $\mathfrak{D}_{S,T}(\varphi) \cup D^t_{S,T}(\varphi) \subseteq D^v_{S,T}(\varphi)$.

d) If $\dim(\varphi) \leq 3$*, then* $\mathfrak{D}_{S,T}(\varphi) \subseteq D_{S,T}(\varphi)$.

e) Let $1 \leq k \leq m$ *be integers,* $\varphi_1, \ldots, \varphi_m$ *be forms over* S *and let* $x_j \in D^v_{S,T}(\varphi_j)$*,* $1 \leq j \leq k$*. Then,*

$$D^v_{S,T}(x_1, \ldots, x_k) \subseteq D^v_{S,T}\left(\bigoplus_{j=1}^m \varphi_j\right).$$

In particular, if ψ *is a* m*-form over* S*, then* $D^v_{S,T}(\varphi) \subseteq D^v_{S,T}(\varphi \oplus \psi)$.

Proof. a) Items $(i) - (iii)$ are clear, while (iv) follows from item (3) in the equivalence in Theorem 1.23 (p. 16) in [**DM2**].

b) The asserted inclusion holds if $\dim \varphi = 2$ (see 2.25). We proceed by induction on $2 \leq n = \dim \varphi$. If $\varphi = \langle b \rangle \oplus \psi$, with $\dim \psi = n$, and $a \in D_{S,T}(\varphi)$, according to 1.7.(d), there are a n-form θ over S and x, y, z_3, $\ldots$, $z_n \in S$ such that $\varphi^T \equiv^S_T \langle a^T \rangle \oplus \theta^T$, $\langle a^T, x^T \rangle \equiv^S_T \langle b^T, y^T \rangle$, $\theta^T \equiv^S_T \langle x^T, z^T_3, \ldots, z^T_n \rangle$ and $\psi^T \equiv^S_T \langle y^T, z^T_3, \ldots, z^T_n \rangle$.

The first isometry above and 2.7.(a.1) entail $a \in D^v_{S,T}(b,\ y)$. Hence, there are $s, t \in T$ such that $a = sb + ty$, while the induction hypothesis yields $y \in D^v_{S,T}(\psi)$. It is now straightforward that $a \in D^v_{S,T}(\varphi)$, completing the induction step.

c) It is clear that $D^t_{S,T}(\varphi) \subseteq D^v_{S,T}(\varphi)$. It remains to check that $\mathfrak{D}_{S,T}(\varphi) \subseteq D^v_{S,T}(\varphi)$. For dim $\varphi = 2$ this holds by definition. Assume the result true for $n \geq 2$ and let $\varphi = \langle\, b\,\rangle \oplus \psi$, where dim $\psi = n$. By items (i) and (iii) in (a), it suffices to check that if $a \in D^v_{S,T}(b,\ u)$, with $u \in D^v_{S,T}(\psi)$, then $a \in D^v_{S,T}(\varphi)$. But we have

$$a = sb + tu \quad \text{and} \quad u = \textstyle\sum_{i=1}^n x_i c_i,$$

where $\psi = \langle\, c_1, \ldots, c_n\,\rangle$ and $s,\, t,\, x_1, \ldots, x_n \in T$; it is immediate from the preceding equalities that $a \in D^v_{S,T}(\varphi)$, as needed.

d) If $\dim(\varphi) = 2$, then $D_{S,T}(\varphi) = D^v_{S,T}(\varphi) = \mathfrak{D}_{S,T}(\varphi)$ and there is nothing to prove. Let $n = 3$. In *any* π-*SG*, G (presently, $G = G_T(S)$), the condition $D_G(b_1,\ b_2,\ b_3) = \bigcup\, \{D_G(b_1,\ u) :\ u \in D_G(b_2,\ b_3)\}$ tells, for $a \in D_G(b_1,\ b_2,\ b_3)$, how to complete $\langle\, a, \cdot, \cdot\,\rangle$ so that it becomes G-isometric to $\langle\, b_1, b_2, b_3\,\rangle$: picking $u \in D_G(b_2,\ b_3)$ so that $a \in D_G(b_1,\ u)$, then $\langle\, a, z\,\rangle \equiv_G \langle\, b_1, u\,\rangle$, where $z = ab_1u$, and $\langle\, b_2, b_3\,\rangle \equiv_G \langle\, u, c\,\rangle$, where $c = b_2b_3u$ ([SG 3]); the third condition required to get $\langle\, a, z, c\,\rangle \equiv_G \langle\, b_1, b_2, b_3\,\rangle$ (cf. 1.7.(d)), namely $\langle\, z, c\,\rangle \equiv_G \langle\, z, c\,\rangle$, holds automatically. Item (e) is straightforward. ■

Note. Without extra assumptions, it is not possible to extend 2.26.(d) to forms of dimension ≥ 4. ■

We set down notation and register some basic facts concerning change of base ring that will be useful in the sequel.

2.27. **Notation and Remarks.** a) Since we shall be dealing simultaneously with several rings, unless clear from the context and to keep notation straight, we adopt the following conventions, where A is a ring and φ, ψ are forms of same dimension with coefficients in $A^\times$:

(1) Write $\varphi \approx^A \psi$ for matrix isometry over A;

(2) If T is a preorder of A, write $\varphi \approx^A_T \psi$ for T-isometry over A.

b) Let $f : A \longrightarrow R$ be a ring morphism and let $n \geq 1$ be an integer. If $M = (m_{ij})$ is a $n \times n$ matrix with coefficients in A, write $f(M) = (f(m_{ij}))$ for the image matrix with coefficients in R. Clearly, if M is diagonal, the same is true of $f(M)$. If M, N are $n \times n$ matrices with coefficients in A, we also have:

(b.1) $\det\ (f(M)) = f(\det M)$; in particular, $M \in \mathrm{GL}_n(A)$ implies $f(M) \in \mathrm{GL}_n(R)$.

(b.2) $f(M)^t = f(M^t)$ and $f(MN) = f(M)\,f(N)$.

c) If $\varphi = \langle\, a_1, \ldots, a_n\,\rangle \in A^n$, set $f \star \varphi = \langle\, f(a_1), \ldots, f(a_n)\,\rangle \in R^n$. Clearly, if φ has coefficients in $A^\times$, then $f \star \varphi$ is a form over $R^\times$. Moreover, if φ, ψ are forms of the same dimension over $A^\times$, then

(c.1) $\varphi \approx^A \psi \quad \Rightarrow \quad f \star \varphi \approx^R f \star \psi$.

(c.2) Let $f : \langle\, A, T\,\rangle \longrightarrow \langle\, R, T'\,\rangle$ be a p-ring morphism, i.e., $f[T] \subseteq T'$. Let $a \in A^\times$. Then,

(i) $\varphi \approx^A_T \psi \quad \Rightarrow \quad f \star \varphi \approx^R_{T'} f \star \psi$;

(ii) $a \in D^v_T(\varphi) \;\;\Rightarrow\;\; f(a) \in D^v_{T'}(f \star \varphi)$ and
$a \in D^t_T(\varphi) \;\;\Rightarrow\;\; f(a) \in D^t_{T'}(f \star \varphi)$.

d) With notation as in (c), let S be a T-subgroup of A and S' be a T'-subgroup of R. If, in addition, $f[S] \subseteq S'$, then the statements corresponding to (c.1) and items (i) and (ii) in (c.2) still hold true in this general setting. ■

CHAPTER 3

The Notion of T-Faithfully Quadratic Ring. Some Basic Consequences

In section 1 of this Chapter we introduce the axioms of T-faithfully quadratic rings and show that if S is T-subgroup of a ring A, where T is A^2 or a preorder of A, and the pair $\langle S, T \rangle$ satisfies these axioms, then the π-SG $G_T(S)$ associated to $\langle S, T \rangle$, is a special group − reduced, in case T is a preorder −, that faithfully codes both value representation and T-isometry of forms of arbitrary dimension over S (Theorem 3.6). We also show that, if the pair $\langle S, T \rangle$ verifies the 2-transversality condition in Lemma 2.7.(b), then the converse is also true (Proposition 3.8). Next, we describe a necessary and sufficient condition guaranteeing that if P is A^2 or a preorder on A, T is a preorder containing P, S is a T-subgroup of A such that $\langle S, P \rangle$ is P-faithfully quadratic, then $\langle S, T \rangle$ is T-faithfully quadratic (Theorem 3.9). We end the section registering that if $\langle A, A^2 \rangle$ is faithfully quadratic, then Milnor's mod 2 K-theory of A is naturally isomorphic to the K-theory of $G(A)$. It will be clear from the following chapters that there is a very significant supply of rings satisfying this property.

In section 2 we discuss, for a T-subgroup S of a p-ring $\langle A, T \rangle$, the relations between orderings on A extending T, signatures on S and T-isometry, showing, in Proposition 3.18, that if $\langle S, T \rangle$ is T-faithfully quadratic, then forms over S are T-isometric iff their global signatures are the same.

1. T-Faithfully Quadratic Rings

We now introduce another notion of fundamental importance in this paper:

DEFINITION 3.1. *Let T be a preorder of a ring A or $T = A^2$. A T-subgroup S of A is* ***T*-faithfully quadratic** *if it verifies the following requirements:*

[T-FQ 1] *: For all $a, b \in S$, $D^v_{S,T}(a, b) = D^t_{S,T}(a, b)$.*

[T-FQ 2] *: For all $n \geq 2$ and all n-forms φ over S,*

$$\mathfrak{D}_{S,T}(\varphi) = D^v_{S,T}(\varphi).$$

[T-FQ 3] *: For all $a \in S$ and all forms φ, ψ over S of the same dimension*

$$\langle a \rangle \oplus \varphi \approx^S_T \langle a \rangle \oplus \psi \;\Rightarrow\; \varphi \approx^S_T \psi. \qquad \blacksquare$$

3.2. **Notation and Remarks** . a) A ring A is ***T*-faithfully quadratic** if the T-subgroup $A^\times$ is T-faithfully quadratic.

b) In case $T = A^2$ we write [FQ i] for [T-FQ i] ($i = 1, 2, 3$), and call **faithfully quadratic** the q-subgroups that verify these conditions. As above, a ring A is **faithfully quadratic** if the q-subgroup $A^\times$ is faithfully quadratic. If $T = \Sigma A^2$, we write [Σ-FQ i] for [T-FQ i] ($i = 1, 2, 3$), and call **Σ-faithfully quadratic** the ΣA^2-subgroups that verify these conditions. Moreover, a ring A is **Σ-faithfully quadratic** if $A^\times$ is Σ-faithfully quadratic.

c) We write $[\text{T-FQ } 2]_m$ or $[\text{T-FQ } 3]_m$ if the statements [T-FQ 2] or [T-FQ 3] are only required to hold for forms of dimension $2 \le n \le m$.

d) Since T-isometry is preserved by scaling, $[\text{T-FQ } 3]_m$ is equivalent to the case where a is any fixed unit (the cases $a = \pm 1$ will be used most frequently).

d) Note that [T-FQ 1] asserts that all binary forms over S are 2-transversal mod T (Lemma 2.7.(b)). ■

We now develop some important consequences of the axioms stated above, starting with the following

THEOREM 3.3. *Let A be a ring and let T be a preorder of A, or $T = A^2$. Let S be a T-subgroup of A and let k, $n \ge 2$ be integers. Assume that S verifies* [T-FQ 1] *and* $[\text{T-FQ } 2]_n$. *Then,*

a) For all $2 \le m \le n$ and all m-forms φ over S, $D^v_{S,T}(\varphi) = D^t_{S,T}(\varphi)$.

b) If $\varphi_1, \ldots, \varphi_k$ are forms over S and $\varphi = \bigoplus_{i=1}^k \varphi_i$ is such that $\dim \varphi \le n$, *then*

$$\begin{aligned} D^v_{S,T}(\varphi) &= \bigcup \{D^v_{S,T}(u_1, \ldots, u_k) : u_i \in D^v_{S,T}(\varphi_i),\ 1 \le i \le k\} \\ &= \bigcup \{D^t_{S,T}(v_1, \ldots, v_k) : v_i \in D^t_{S,T}(\varphi_i),\ 1 \le i \le k\}. \end{aligned}$$

PROOF. a) By [T-FQ 1], the result is true for $m = 2$. We proceed by induction on m, recalling that, by 2.26.(c), it suffices to verify that $D^v_{S,T}(\varphi) \subseteq D^t_{S,T}(\varphi)$. Let $\varphi = \langle a_1 \rangle \oplus \psi$, with $m = \dim \psi < n$ and let $x \in D^v_{S,T}(\varphi)$. By [T-FQ 1], $[\text{T-FQ } 2]_n$ and the induction hypothesis, there is $u \in D^v_{S,T}(\psi) = D^t_{S,T}(\psi)$ such that $x \in D^v_{S,T}(a_1, u)$ $= D^t_{S,T}(a_1, u)$. It is now straightforward to see that $x \in D^t_{S,T}(\varphi)$. Indeed, if $\psi = \langle c_1, \ldots, c_m \rangle$, there are $s, t, z_1, \ldots, z_m \in T^\times$, such that $x = sa_1 + tu$, with $u = \sum_{i=1}^m z_i c_i$, as needed.

b) It suffices to verify the first equality for $k = 2$; a straightforward induction will complete its proof, while the second follows from (a) and the fact that $k \le n$ and $\max_{1 \le i \le k} (\dim \varphi_i) \le n$. Moreover, we may assume that $n \ge 3$, otherwise there is nothing to prove. Suppose, then, that $\varphi = \varphi_1 \oplus \varphi_2$; by $[\text{T-FQ } 2]_n$ the result is true if $\dim \varphi_1 = 1$. We proceed by induction on $m = \dim \varphi_1 < n$, letting $\varphi_1 = \langle a_1 \rangle \oplus \psi$, with $\dim \psi = m$. Fix $a \in D^v_{S,T}(\varphi)$; by $[\text{T-FQ } 2]_n$, there is $x \in D^v_{S,T}(\psi \oplus \varphi_2)$ such that

(I) $$a \in D^v_{S,T}(a_1, x).$$

Since $\dim \psi = m$, the induction hypothesis yields $u \in D^v_{S,T}(\psi)$ and $v \in D^v_{S,T}(\varphi_2)$ such that

(II) $$x \in D^v_{S,T}(u, v).$$

It follows from (I) and (II) that $a \in D^v_{S,T}(a_1, u, v)$. Because $n \ge 3$, we have

$$D^v_{S,T}(a_1, u, v) = \mathfrak{D}_{S,T}(a_1, u, v) \subseteq \bigcup \{D^v_{S,T}(z, v) : z \in D^v_{S,T}(a_1, u)\},$$

hence there is $z \in D^v_{S,T}(a_1,\, u)$ so that $a \in D^v_{S,T}(z,\, v)$. Since $u \in D^v_{S,T}(\psi)$, 2.26.(e) entails $z \in D^v_{S,T}(\langle\, a_1 \,\rangle \oplus \psi) \;=\; D^v_{S,T}(\varphi_1)$ and so $a \in D^v_{S,T}(z,\, v)$, with $z \in D^v_{S,T}(\varphi_1)$ and $v \in D^v_{S,T}(\varphi_2)$, completing the induction step. ■

LEMMA 3.4. *Let A be a ring and let T be a preorder of A, or $T = A^2$. Let S be a T-subgroup of A and let $\varphi = \langle\, a_1, \dots, a_n \,\rangle$, ψ be S-forms of the same dimension n. Then,*

$$\varphi \approx^S_T \psi \quad \Rightarrow \quad \{a_1, \dots, a_n\} \subseteq D^v_{S,T}(\psi).$$

PROOF. If $\langle\, c_1, \dots, c_n \,\rangle$ and $\theta = \langle\, d_1, \dots, d_n \,\rangle$ are S-forms, then, with notation as in Definition 2.24, we have

$$\text{(I)} \qquad \langle\, c_1, \dots, c_n \,\rangle \approx \theta \quad \Rightarrow \quad \{c_1, \dots, c_n\} \subseteq D^v_{S,T}(\theta).$$

Indeed, if $M \in \mathrm{GL}_n(A)$ is such that $\mathscr{M}(c_1, \dots, c_n) = M\mathscr{M}(\theta)M^t$, then $c_j = \sum_{i=1}^n x_{ji}^2 d_i$, where $(x_{j1},\, \dots,\, x_{jn})$ is the j^{th}-row of M, $1 \le j \le n$. Since $A^2 \subseteq T$, we have $D^v_{S,A^2}(\theta) \subseteq D^v_{S,T}(\theta)$, and (I) follows immediately.

Assume that $\varphi = \langle\, a_1, \dots, a_n \,\rangle \approx^S_T \psi$; since for all permutations σ of $\{1, \dots, n\}$, $\varphi^\sigma \approx^S_T \varphi$, it suffices to show that $a_1 \in D^v_{S,T}(\psi)$. Let $\varphi_0, \varphi_1, \dots, \varphi_k$ be a sequence of n-dimensional S-forms witnessing $\varphi \approx^S_T \psi$. By induction on $0 \le i \le k$, we show that $a_1 \in D^v_{S,T}(\varphi_i)$, a fact that is clear for $i = 0$. Assume the induction hypothesis holds for $i < k$; we have two possibilities:

∗ If $\varphi_i \approx \varphi_{i+1}$, it follows from (I) that all entries of φ_i are in $D^v_{S,T}(\varphi_{i+1})$. Since $a_1 \in D^v_{S,T}(\varphi_i)$, elementary computations show: $a_1 \in D^v_{S,T}(\varphi_{i+1})$;

∗ If $\varphi_{i+1} = \langle\, t_1x_1, \dots, t_nx_n \,\rangle$, $\varphi_i = \langle\, x_1, \dots, x_n \,\rangle$, with $\{t_1, \dots, t_n\} \subseteq T^\times$, let $s_1, \dots, s_n \in T$ be such that $a_1 = \sum_{i=1}^n s_i x_n$. Then, $a_1 = \sum_{i=1}^n (s_i/t_i)\, t_i x_i$, completing the induction step and the proof. ■

THEOREM 3.5. *Let A be a ring and T be a preorder of A or $T = A^2$. If S is a T-subgroup of A or a q-subgroup of A, then*

a) With notation as in 2.24 *and* 3.3, *assume that value representation of S-forms verifies* [T-FQ 2]$_3$. *If ψ is a S-form of dimension ≤ 3, then $D^v_{S,T}(\psi) \;=\; D_{S,T}(\psi)$, that is, an element of S is value represented mod T iff it is isometry represented by ψ^T in $G_T(S)$.*

b) Assume that value representation of S-forms verifies [T-FQ 1], [T-FQ 2]$_3$ *and* [T-FQ 3]$_3$. *Then,*

(1) *For all 3-forms φ, ψ over S, $\varphi \approx^S_T \psi \;\Leftrightarrow\; \varphi^T \equiv^S_T \psi^T$;*

(2) *$G_T(S) \;=\; \langle\, S/T^\times, \equiv^S_T, -1 \,\rangle$ is a special group, which is reduced if T is a preorder.*

PROOF. a) Lemma 2.26.(b), (d) gives: $\mathfrak{D}_{S,T}(\varphi) \subseteq D_{S,T}(\varphi) \subseteq D^v_{S,T}(\varphi)$. Hence, in the presence of [T-FQ 2]$_3$, all three sets are equal.

b) (1) By 2.22, it suffices to prove ($\Rightarrow$). Let $\varphi = \langle\, a, x, y \,\rangle$ and $\psi = \langle\, b_1, b_2, b_3 \,\rangle$. By 3.4, $\varphi \approx^S_T \psi$ implies $a \in D^v_{S,T}(\psi) = \mathfrak{D}_{S,T}(\psi)$. Setting $z = ab_1u$ and $c = b_2b_3u$, where $u \in D^v_{S,T}(b_2,\, b_3)$, the proof of Lemma 2.26.(d) gives:

(I) $\langle a^T, c^T, z^T \rangle \equiv_T^S \langle b_1^T, b_2^T, b_3^T \rangle$ and

(II) $\langle a^T, z^T \rangle \equiv_T^S \langle b_1^T, u^T \rangle$, $\langle b_2^T, b_3^T \rangle \equiv_T^S \langle u^T, c^T \rangle$.

By 2.22, the isometry (I) entails $\langle a, c, z \rangle \approx_T^S \langle b_1, b_2, b_3 \rangle$. Since $\approx_T^S$ is transitive, this relation and our hypothesis yield $\langle a, c, z \rangle \approx_T^S \langle a, x, y \rangle$. Now, $[\text{T-FQ } 3]_3$ entails $\langle c, z \rangle \approx_T^S \langle x, y \rangle$, whence, by Lemma 2.21.(a), $\langle x^T, y^T \rangle \equiv_T^S \langle z^T, c^T \rangle$. This last isometry, together with those in (II) above yield

$$\langle a^T, z^T \rangle \equiv_T^S \langle b_1^T, u^T \rangle, \; \langle b_2^T, b_3^T \rangle \equiv_T^S \langle u^T, c^T \rangle \quad \text{and}$$
$$\langle x^T, y^T \rangle \equiv_T^S \langle z^T, c^T \rangle,$$

that is equivalent to $\langle a^T, x^T, y^T \rangle \equiv_T^S \langle b_1^T, b_2^T, b_3^T \rangle$, as desired. Remark that [T-FQ 1] is used in Lemma 2.21.

(2) As observed in Lemma 2.7.(b), 2-transversality (i.e., [T-FQ 1]) guarantees that $G_T(S)$ is a pre-special group. To be a special group it suffices that the isometry relation $\equiv_T^S$ be transitive for 3-forms. Since T-isometry is transitive, this is an immediate consequence of (b.1). The reducedness of $G_T(S)$ if T is a preorder was observed in 2.7.(c). ■

THEOREM 3.6. *Let A be a ring and let T be a preorder of A or $T = A^2$. Let S be a T-subgroup of A. Assume that S satisfies* [T-FQ 1], [T-FQ 2] *and* $[\text{T-FQ } 3]_3$. *Then,*

a) For all S-forms, φ, $D_{S,T}(\varphi) = D^v_{S,T}(\varphi)$, that is, an element of S is value represented iff it is isometry represented in $G_T(S)$.

If, in addition, S satisfies [T-FQ 3] *—i.e., S is T-faithfully quadratic— then,*

b) For all S-forms φ, ψ of the same dimension,

$$\varphi \approx_T^S \psi \quad \Leftrightarrow \quad \varphi^T \equiv_T^S \psi^T.$$

c) $G_T(S) = \langle S/T^\times, \equiv_T^S, -1 \rangle$ is a special group that, according to (a) and (b) above, faithfully represents T-isometry and value representation of diagonal quadratic forms with entries in S [1]. *Moreover, $G_T(S)$ is reduced if T is a preorder.*

PROOF. By Theorem 3.5, we know that $G_T(S)$ is a special group. By Lemma 1.8.(f) (see also Theorem 1.23 in [**DM2**]), for all $n \geq 2$, the extension of $\equiv_T^S$ to forms of dimension n *is a transitive relation.* We shall use this below, without further comment.

a) By Lemma 2.26.(b) it is enough to verify that $D^v_{S,T}(\varphi) \subseteq D_{S,T}(\varphi)$, which will be achieved by induction on $\dim \varphi \geq 2$. It follows from 3.5.(a) that the result holds true for $\dim \varphi \leq 3$. Assume it valid for forms of dimension n and let $\varphi = \langle b \rangle \oplus \psi$, where $\dim \psi = n$. If $a \in S$ is value-represented by $\langle b \rangle \oplus \psi$, then $[\text{T-FQ } 2]_n$ implies that there is $u \in S$ such that

(I) $$a \in D^v_{S,T}(b, u) \quad \text{and} \quad u \in D^v_{S,T}(\psi).$$

The induction hypothesis yields $z_2, \ldots, z_n \in S$ such that

(II) $$\langle u^T, z_2^T, \ldots, z_n^T \rangle \equiv_T^S \psi^T,$$

[1] See also 3.7 and Proposition 3.8.

while the first representation relation in (I) implies $\langle\, a^T,\, (abu)^T \,\rangle \equiv_T^S \langle\, b^T,\, u^T \,\rangle$. Adding $\langle\, b^T \,\rangle$ to both sides of (II), 1.13.(c) yields

$$\text{(III)} \qquad \langle\, b^T,\, u^T,\, z_2^T, \ldots, z_n^T \,\rangle \equiv_T^S \langle\, b^T \,\rangle \oplus \psi^T.$$

Since $\langle\, a^T,\, (abu)^T \,\rangle \equiv_T^S \langle\, b^T,\, u^T \,\rangle$, another application of 1.13.(c) yields

$$\text{(IV)} \qquad \langle\, a^T,\, (abu)^T,\, z_2^T, \ldots, z_n^T \,\rangle \;\equiv_T^S\; \langle\, b^T,\, u^T,\, z_2^T, \ldots, z_n^T \,\rangle.$$

Now, (III), (IV) and the transitivity of $\equiv_T^S$ entail

$$\langle\, a^T,\, (abu)^T,\, z_2^T, \ldots, z_n^T \,\rangle \;\equiv_T^S\; \langle\, b^T \,\rangle \oplus \psi^T,$$

wherefrom we conclude $a \in D_{S,T}(\langle\, b \,\rangle \oplus \psi)$, as needed.

b) By Corollary 2.22, it is enough to prove the implication ($\Rightarrow$), that we know, by 3.5.(b.1), to hold for forms of dimension ≤ 3. We proceed by induction on dimension; assume the result holds for forms of dimension n and suppose $\varphi = \langle\, a \,\rangle \oplus \theta_1$, $\psi = \langle\, b \,\rangle \oplus \theta_2$, where $\dim \theta_i = n$ ($i = 1, 2$). If $\varphi \approx_T^S \psi$, then 3.4 entails $a \in D_{S,T}^v(\psi)$ and so, as above, $[\text{T-FQ } 2]_n$ and item (a) yield $u, z_2, \ldots, z_n \in S$ such that

$$\text{(V)} \qquad \langle\, a^T,\, (abu)^T \,\rangle \equiv_T^S \langle\, b^T,\, u^T \,\rangle \quad \text{and} \quad \langle\, u^T,\, z_2^T, \ldots, z_n^T \,\rangle \;\equiv_T^S\; \theta_2^T.$$

Adding $\langle\, b^T \,\rangle$ to both sides of the second isometry in (V) gives

$$\text{(VI)} \qquad \langle\, b^T,\, u^T,\, z_2^T, \ldots, z_n^T \,\rangle \;\equiv_T^S\; \psi^T.$$

On the other hand, by adding $\langle\, z_2^T, \ldots, z_n^T \,\rangle$ to both sides, the first isometry in (V) entails,

$$\langle\, a^T,\, (abu)^T,\, z_2^T, \ldots, z_n^T \,\rangle \;\equiv_T^S\; \langle\, b^T,\, u^T,\, z_2^T, \ldots, z_n^T \,\rangle,$$

that together with (VI) and the transitivity of $\equiv_T^S$ implies

$$\text{(VII)} \qquad \langle\, a^T,\; (abu)^T,\, z_2^T, \ldots, z_n^T \,\rangle \;\equiv_T^S\; \psi^T.$$

Corollary 2.22 and the assumption $\varphi \approx_T^S \psi$ give:

$$\langle\, a, abu, z_2, \ldots, z_n \,\rangle \;\approx_T^S\; \psi \;\approx_T^S\; \varphi \;=\; \langle\, a \,\rangle \oplus \theta_1.$$

Now, using [T-FQ 3] we can cancel out $\langle\, a \,\rangle$ to get $\langle\, abu, z_2, \ldots, z_n \,\rangle \approx_T^S \theta_1$. Since $\dim \theta_1 = n$, the induction hypothesis applies, to yield $\langle\, (abu)^T,\, z_2^T, \ldots, z_n^T \,\rangle \equiv_T^S \theta_1^T$, wherefrom, adding $\langle\, a^T \,\rangle$ to both sides and using (VII) we get

$$\psi^T \;\equiv_T^S\; \langle\, a^T, (abu)^T, z_2^T, \ldots, z_n^T \,\rangle \;\equiv_T^S\; \langle\, a^T \,\rangle \oplus \theta_1^T \;=\; \varphi^T,$$

as required. Item (c) follows immediately from (b) and 3.5.(b). ∎

3.7. **Discussion.** The arguments that follow suggest that the axioms [T-FQ i] ($i = 1, 2, 3$) presented in 3.1 above are natural. Let S be a q-subgroup of a ring A.

(1) The hypothesis of 2-transversality for representation in $G_T(S)$, i.e., [T-FQ 1] in 3.1, seems to be crucial to show that $G_T(S)$ satisfies [SG 4] (see Lemma 2.7). Note that [SG 4] *entails* [SG 2] and so one might be tempted to omit the requirement that $2 \in A^\times$, used to establish [SG 2]. However, the hypothesis that $2 \in A^\times$ has other useful consequences (e.g., a preorder T is proper iff $-1 \notin T$; cf. also 4.9) and guarantees that all residue fields of A modulo maximal ideals are of characteristic $\neq 2$, a classical setting for quadratic form theory.

(2) With notation as in 2.4, what would be reasonable requirements on $G_T(S) = \langle S/T^\times, \equiv_T^S, -1 \rangle$ so that it **faithfully** represents the theory of diagonal S-quadratic forms mod T? The first would be that isometry in $G_T(S)$ corresponds to $\approx_T^S$: if φ, ψ are (diagonal) S-quadratic forms of the same dimension, then

$$(*) \qquad \varphi \approx_T^S \psi \quad \Leftrightarrow \quad \varphi^T \equiv_T^S \psi^T.$$

Next, since value-representation is another essential ingredient in quadratic form theory, it would be natural to expect that it corresponds to representation in $G_T(S)$, that is, for all $a \in S$ and all S-forms φ

$$(**) \qquad a \in D^v_{S,T}(\varphi) \quad \Leftrightarrow \quad a^T \in D(\varphi^T) \text{ in } G_T(S).$$

With notation as in Theorems 3.3, 3.5 and 3.6, we have

PROPOSITION 3.8. *Let A be a ring and let T be a preorder of A, or $T = A^2$. Let S be a T-subgroup of A satisfying* [T-FQ 1]. *Then, the following are equivalent:*

(1) *$G_T(S)$ is a RSG such that for all S-forms of the same dimension, φ, ψ,*

$(*)$ $\varphi \approx_T^S \psi \Leftrightarrow \varphi^T \equiv_T^S \psi^T$; $\qquad (**)$ $D^v_{S,T}(\varphi) = D_{S,T}(\varphi)$.

(2) *S and T satisfy conditions* [T-FQ 2] *and* [T-FQ 3].

PROOF. (2) $\Rightarrow$ (1) is the content of Theorems 3.5 and 3.6. For the converse, first note that, since any special group satisfies Witt-cancellation (Proposition 1.6.(b), [**DM2**]), (*) immediately yields [T-FQ 3].

To prove [T-FQ 2], let $\varphi = \langle b_1, \ldots, b_n \rangle$ be a n-ary S-form. By 2.26.(c), it suffices to check that $D^v_{S,T}(\varphi) \subseteq \mathfrak{D}_{S,T}(\varphi)$. Assume that, for $a \in S$ we have $a \in D^v_{S,T}(\varphi)$; then (**) yields $a^T \in D(\varphi^T)$ in $G_T(S)$ (cf. Remark 2.25), and hence (2.24.(a)), there is a S-form θ such that $\langle a^T \rangle \oplus \theta^T \equiv_T^S \varphi^T$.

Fix $1 \leq k \leq m$. Since $G_T(S)$ is a special group, Theorem 1.23.(3) in [**DM2**] yields

$$\varphi^T \equiv_T^S \langle b_k^T \rangle \oplus \langle b_1^T, \ldots, \check{b}_k^T, \ldots, b_n^T \rangle,$$

and

$$\langle a^T \rangle \oplus \theta^T \equiv_T^S \langle b_k^T \rangle \oplus \langle b_1^T, \ldots, \check{b}_k^T, \ldots, b_n^T \rangle.$$

Hence,

$$a^T \in D_{S,T}\left(\langle b_k^T \rangle \oplus \langle b_1^T, \ldots, \check{b}_k^T, \ldots, b_n^T \rangle \right).$$

By Proposition 1.6.(c) in [**DM2**], there is $u \in S$ such that

$$a^T \in D(b_k^T, u^T) \quad \text{and} \quad u^T \in D(b_1^T, \ldots, \check{b}_k^T, \ldots, b_n^T) \quad \text{in } G_T(S).$$

Now, the equivalence (I) in 2.25 and assumption (**) entail

$$a \in D^v_{S,T}(b_k, u) \quad \text{and} \quad u \in D^v_{S,T}(b_1, \ldots, \check{b}_k, \ldots, b_n),$$

as needed. ■

Let P be A^2 or a preorder of a ring A and let T be a preorder, with $P \subseteq T$. Let S be a T-subgroup of A. Item (b) of our next result gives a useful and simple necessary and sufficient condition for the quadratic faithfulness of S to lift from P to T.

Theorem 3.9. *Let A be a ring, let $P = A^2$ or a proper preorder of A and let T be a proper preorder of A containing P. Let S be a T-subgroup of A that is P-faithfully quadratic. Then,*

a) $\Delta_T = \{a^P \in G_P(S) : a \in T^\times\} = q_P[T]$ *is a proper saturated subgroup of* $G_P(S)$. *Moreover, if S verifies* [T-FQ 1], *the map*

$$\gamma : G_T(S) = S/T^\times \longrightarrow G_P(S)/\Delta_T,$$

given by $\gamma(a^T) = a^P/\Delta_T$ *is the* unique isomorphism *of π-RSGs, making the following diagram commute:*

$$\begin{array}{ccc} S & \xrightarrow{\;q_T\;} & G_T(S) \\ {\scriptstyle q_P}\big\downarrow & & \big\downarrow{\scriptstyle \gamma} \\ G_P(S) & \xrightarrow[\;can.\;]{} & G_P(S)/\Delta_T \end{array}$$

where q_P, q_T and can. are the natural quotient maps. In particular, $G_T(S)$ is a RSG.

b) The following are equivalent:

(1) *S is T-faithfully quadratic;*

(2) *For all $x, a_1, \ldots, a_n \in S$, if $x \in D^v_{S,T}(a_1, \ldots, a_n)$, then there are $x_2, \ldots, x_n \in S$ and $t_1, \ldots, t_n \in T^\times$ such that $\langle x, x_2, \ldots, x_n \rangle \approx^S_P \langle t_1a_1, \ldots, t_na_n \rangle$;*

(3) *For all S-forms φ, $D^v_{S,T}(\varphi) = D^t_{S,T}(\varphi)$, i.e., representation of S-forms mod T is transversal.*

Remark. By Theorem 3.3.(a), the condition

$$\text{For all } S\text{-forms, } \varphi, \quad D^v_{S,T}(\varphi) = D^t_{S,T}(\varphi),$$

is necessary for S to be T-faithfully quadratic; Theorem 3.9 guarantees its sufficiency. ■

Proof of Theorem 3.9. The notational convention set in 2.25 will be used throughout this proof.

a) Clearly, Δ_T is a subgroup of $G_P(S)$ and $-1 \notin \Delta_T$, otherwise -1 would be in T. To show it is saturated (1.11), let $a, b \in S$ be such that

(I) $$a^P \in D_{S,P}(1, b^P) \quad \text{and} \quad b^P \in \Delta_T.$$

The first statement in (I) gives $a \in D^v_{S,P}(1, b)$ (by 2.7.(a.1)), while the second yields $t \in T^\times$ and $c \in P^\times$ with $b = tc \in T^\times$. Hence, there are $d_1, d_2 \in P$, such that $a = d_1 + d_2b$; since $P \subseteq T$ we get $a \in T^\times$ and $a^P \in \Delta_T$, establishing its saturation.

By Proposition 2.21 (p. 42) of [**DM2**], $G_P(S)/\Delta_T$ is a RSG, which will be written H. Since $P^\times \subseteq T^\times$, $G_P(S) = S/P^\times$, $\Delta_T = T^\times/P^\times$ and $G_T(S) = S/T^\times$, the fundamental theorem of morphism of groups, yields a unique *group isomorphism*, γ, making the displayed diagram commutative. Clearly, γ takes -1 in $G_T(S)$ to

-1 in H. To see that γ is an isomorphism of π-RSGs, it remains to check, with notation as in 2.25, that for a, $b \in S$ we have

$$a^T \in D(1, b^T) \text{ in } G_T(S) \;\Leftrightarrow\; a^P/\Delta_T \in D_H(1,\; b^P/\Delta_T).$$

($\Rightarrow$) : If $a^T \in D(1, b^T)$, then $a \in D^v_{S,T}(1, b)$ (cf. 2.7.(a.1)); since S verifies [T-FQ 1], there are s, $t \in T^\times$ such that $a = s + tb$, which may be rewritten as $xa = 1 + yb$, with $x = 1/s$, $y = t/s \in T^\times$. Hence, $xa \in D^v_{S,T}(1, yb)$, and 2.7.(a.1) yields $(xa)^T \in D(1, (yb)^T)$ in $G_T(S)$. Thus, Proposition 2.21 of [**DM2**] yields, $a^P/\Delta_T \in D_H(1,\; b^P/\Delta_T)$, because $x \in T^\times$ entails $(xa)^P/\Delta_T = a^P/\Delta_T$.

($\Leftarrow$) : Assume $a^P/\Delta_T \in D_H(1,\; b^P/\Delta_T)$; Def. 2.13 (p. 39) and Prop. 2.28 (p. 45) of [**DM2**] yield s, $t \in T^\times$ such that $(sa)^P \in D(1, (tb)^P)$ in $G_P(S)$. As above, we get $sa \in D^v_{S,P}(1, tb)$, which implies, as $P \subseteq T$, $a \in D^v_{S,T}(1, b)$ and so $a^T \in D(1, b^T)$, as required.

Note that no assumption on S is required to prove ($\Leftarrow$); thus, the inverse of γ is *always* a morphism of π-RSGs.

b) (1) $\Rightarrow$ (2) : Let $\psi = \langle a_1, \ldots, a_n \rangle$; since S is T-faithfully quadratic, Theorem 3.3.(a) yields $x \in D^v_{S,T}(\psi) = D^t_{S,T}(\psi)$, and so there are $t_1, \ldots, t_n \in T^\times$ such that $x = \sum_{j=1}^n t_j a_j$. Hence, $x \in D^v_{S,P}(t_1a_1, \ldots, t_na_n)$ and the P-quadratic faithfulness of S entails $x^P \in D(t_1^P a_1^P, \ldots, t_n^P a_n^P)$ in the special group $G_P(S)$. The definition of representation in $G_P(S)$ furnishes $x_2, \ldots, x_n \in S$ such that $\langle x^P, x_2^P, \ldots, x_n^P \rangle \equiv^S_P \langle t_1^P a_1^P, \ldots, t_n^P a_n^P \rangle$, which in turn implies, by 2.22, $\langle x, x_2, \ldots, x_n \rangle \approx^S_P \langle t_1a_1, \ldots, t_na_n \rangle$.

(2) $\Rightarrow$ (3) : Let $\varphi = \langle a_1, \ldots, a_n \rangle$; if $x \in D^v_{S,T}(\varphi)$, using assumption (2), let $t_1, \ldots, t_n \in T^\times$ and $x_2, \ldots, x_n \in S$ be such that $\langle x, x_2, \ldots, x_n \rangle \approx^S_P \langle t_1a_1, \ldots, t_na_n \rangle$. By 3.4, we have $x \in D^v_{S,P}(t_1a_1, \ldots, t_na_n)$ and the P-quadratic faithfulness of S and Theorem 3.3(a) yield

$$x \in D^v_{S,P}(t_1a_1, \ldots, t_na_n) \;=\; D^t_{S,P}(t_1a_1, \ldots, t_na_n).$$

Thus, there are $c_1, \ldots, c_n \in P^\times$ such that $x = \sum_{j=1}^n c_j t_j a_j$, whence, since $P^\times \subseteq T^\times$, $x \in D^t_{S,T}(\varphi)$, as needed.

It remains to show that (3) $\Rightarrow$ (1), to be accomplished through Facts 3.10 and 3.11, below. Assume that S satisfies (3); in particular, S satisfies [T-FQ 1].

Fact 3.10. [T-FQ 2] *holds in* S.

Proof. By Lemma 2.26.(c), it suffices to check that if $\varphi = \langle a_1, \ldots, a_n \rangle$ is an S-form with $n \geq 3$, then $D^v_{S,T}(\varphi) \subseteq \mathfrak{D}_{S,T}(\varphi)$. Fix $1 \leq k \leq n$ and $b \in D^v_{S,T}(\varphi) = D^t_{S,T}(\varphi)$. Then, there are $t_1, \ldots, t_n \in T^\times$ such that

$$\text{(II)} \qquad b \;=\; t_k a_k + \textstyle\sum_{i \neq k} t_i a_i.$$

Let $\theta = \langle t_1a_1, \ldots, \check{(t_ka_k)}, \ldots, t_na_n \rangle$. Then, (II) implies

$$b \in D^v_{S,P}(\langle t_ka_k \rangle \oplus \theta) \qquad (P\text{-representation}),$$

and [P-FQ 2] yields $u \in S$ such that $b \in D^v_{S,P}(t_ka_k,\; u)$, with $u \in D^v_{S,P}(\theta)$. By Theorem 3.3.(a), there are $d, c_1, \ldots, c_n$ in $P^\times \subseteq T^\times$ such that

$$b = c_k t_k a_k + du \quad \text{and} \quad u = \textstyle\sum_{i \neq k} c_i t_i a_i,$$

and $b \in D^v_{S,T}(a_k, u)$, with $u \in D^v_{S,T}(a_1, \ldots, \check{a}_k, \ldots, a_n)$, as required. □

FACT 3.11. [T-FQ 3] *holds in* S.

PROOF. Let $n \geq 2$ be an integer. Let $a \in S$, let φ, ψ be $(n-1)$-dimensional forms over S, and let $\varphi_0, \varphi_1, \ldots, \varphi_k$ be a sequence of n-dimensional S-forms witnessing $\langle a \rangle \oplus \varphi \approx^S_T \langle a \rangle \oplus \psi$. Consider the sequence of $G_P(S)$-forms $\varphi^P_0, \ldots, \varphi^P_k$; clearly, $\varphi^P_0 = \langle a^P \rangle \oplus \varphi^P$ and $\varphi^P_k = \langle a^P \rangle \oplus \psi^P$. We claim that for all $1 \leq i \leq k$,

$$\text{(III)} \qquad \begin{cases} (i) \quad \varphi^P_i \equiv^S_P \varphi^P_{i-1} \quad \text{or} \\ (ii)\ \varphi^P_i = \langle (t_1x_1)^P, \ldots, (t_nx_n)^P \rangle, \text{ with} \\ \qquad \varphi^P_{i-1} = \langle x^P_1, \ldots, x^P_n \rangle \text{ and } t_i \in T^\times. \end{cases}$$

Indeed, for each $1 \leq i \leq n$:

- If $\varphi_i \approx \varphi_{i-1}$, since S is P-faithfully quadratic, item (b) in 3.6 entails $\varphi^P_i \equiv^S_P \varphi^P_{i-1}$;
- If $\varphi_i = \langle t_1x_1, \ldots, t_nx_n \rangle$, with $\varphi_{i-1} = \langle x_1, \ldots, x_n \rangle$, we are clearly in alternative (ii) of (III),

as claimed. By item (a), we may identify $G_T(S)$ with $G_P(S)/\Delta_T$, via the isomorphism γ. Now consider the sequence $\varphi^T_0, \ldots, \varphi^T_k$ of n-forms in $G_T(S)$; again, $\varphi^T_0 = \langle a^T \rangle \oplus \varphi^T$, $\varphi^T_k = \langle a^T \rangle \oplus \psi^T$, and we have

$$\text{(IV)} \qquad \text{For all } 1 \leq i \leq k, \quad \varphi^T_0 \equiv^S_T \varphi^T_i.$$

Indeed, this is clear if we are in alternative (ii) of (III) since $\varphi^T_i = \varphi^T_{i-1}$. If (i) in (III) holds, since the natural quotient map from $G_P(S)$ to $G_T(S)$ is a π-SG morphism, we get $\varphi^T_i \equiv^S_T \varphi^T_{i-1}$. Now, by (a), $G_T(S)$ is a RSG and so $\equiv^S_T$ is transitive for forms of every dimension, wherefrom (IV) follows immediately. In particular, we have $\langle a^T \rangle \oplus \varphi^T \equiv^S_T \langle a^T \rangle \oplus \psi^T$. Since $G_T(S)$ is a SG, Witt cancellation (Proposition 1.6.(b) (p.4) in [**DM2**]) yields $\varphi^T \equiv^S_T \psi^T$ and 2.22 guarantees that $\varphi \approx^S_T \psi$, completing the proof of Theorem 3.9. ■

Often, [T-FQ 3] is the hardest axiom to verify in concrete examples. The following criterion can profitably be used to facilitate this task.

PROPOSITION 3.12. *Let* $f : \langle A, T \rangle \longrightarrow \langle R, P \rangle$ *be a p-ring morphism. With notation as in* 2.8, *let* $f^\pi : G_T(A) \longrightarrow G_P(R)$, $f^\pi(a^T) = f(a)^P$, *be the* π*-RSG morphism induced by* f. *If* $\langle R, P \rangle$ *is* P*-faithfully quadratic and* f^π *is a complete embedding* (*cf.* 1.14.(*c*)), *then* $\langle A, T \rangle$ *satisfies* [T-FQ 3]. *Hence, if* $\langle A, T \rangle$ *verifies* [T-FQ 1] *and* [T-FQ 2], *then* $\langle A, T \rangle$ *is* T*-faithfully quadratic.*

PROOF. Let $a \in A^\times$ and φ, ψ be forms over $A^\times$ such that $\langle a \rangle \oplus \varphi \approx^A_T \langle a \rangle \oplus \psi$; then, $\langle f(a) \rangle \oplus f \star \varphi \approx^R_P \langle f(a) \rangle \oplus f \star \psi$ (cf. 2.27.(c.2.(i))). The P-quadratic faithfulness of $\langle R, P \rangle$ and 3.6.(c) entail $\langle f(a)^P \rangle \oplus (f \star \varphi)^P \equiv^R_P \langle f(a)^P \rangle \oplus (f \star \psi)^P$. Since $G_P(R)$ is a SG, Witt cancellation (Proposition 1.6.(b) in [**DM2**]) implies $(f \star \varphi)^P \equiv^R_P (f \star \psi)^P$, that is, $f^\pi \star \varphi^T \equiv^R_P f^\pi \star \psi^T$. Since f^π is a complete embedding, we obtain $\varphi^T \equiv^A_T \psi^T$, and Corollary 2.22 yields $\varphi \approx^A_T \psi$, showing that $\langle A, T \rangle$ verifies [T-FQ 3]. The last assertion in the statement follows immediately. ■

The following is an interesting consequence of the preceding discussion and of results to be proven in Chapter 7.

Recall that a ring A is **Pythagorean** if $A^2 = \Sigma A^2$. Moreover, A is said to have **weak bounded inversion** if $1 + \Sigma A^2 \subseteq A^\times$.

COROLLARY 3.13. *If A is a Pythagorean ring with weak bounded inversion or a faithfully quadratic ring, then k_*A is naturally isomorphic to $k_*G(A)$.*

PROOF. By Theorem 7.6.(d) every Pythagorean ring with weak bounded inversion verifies the transversality principle [FQ 1] and the conclusion follows from Theorem 2.16. The assertion concerning faithfully quadratic rings is clear. ∎

2. Orders, Signatures and T-isometry

If $\langle A, T \rangle$ is a p-ring, and S is a T-subgroup of A, we first show that a natural concept of signature on S, essentially due to Knebusch, Rosenberg and Ware (see [**KRW**]), coincides with that of SG-characters defined on the π-SG associated to S. We then discuss the question of whether signatures on S come from orderings on A –i.e., elements of the real spectrum of A– containing T. We show that if S verifies axiom [T-FQ 2], then this is indeed the case.[2] We end this section by proving that if S is T-faithfully quadratic, then an analog of Pfister's classical local-global principle in the reduced theory of quadratic forms over a field extends to the present setting.

Recall (1.10.(d)) that $\mathbb{Z}_2 = \{1, -1\}$ is the two-element reduced special group and that the definition of SG-character appears in 1.12.(b). Following the lead of [**KRW**], we set down the following

DEFINITION 3.14. *Let $\langle A, T \rangle$ be a proper p-ring and let S be T-subgroup of A.*

(1) *A **T-signature** on S is a group morphism, $\tau : S \longrightarrow \mathbb{Z}_2$, such that $\tau(-1) = -1$ and for all $a \in S$,*

$$a \in \ker \tau \quad \Rightarrow \quad D^v_{S,T}(1, a) \subseteq \ker \tau.$$

*Write $Z_{S,T}$ for the set of T-signatures on S. A **T-signature** on A is a T-signature on $A^\times$ and we write $Z_{A,T}$ for $Z_{A^\times,\,T}$.*

(2) *If $\varphi = \langle a_1, \ldots, a_n \rangle$ is a form over S and $\tau \in Z_{S,T}$, the* integer $sgn_\tau(\varphi) = \sum_{i=1}^n \tau(a_i)$ *is the* **signature of φ at τ**. ∎

Note that if $\tau \in Z_{S,T}$, since $T^\times \subseteq D^v_{S,T}(1,\,1)$, we have $T^\times \subseteq \ker \tau$.

EXAMPLE 3.15. Let $\langle A, T \rangle$ be a proper p-ring and let S be a T-subgroup of A; let $Y_T = \mathrm{Sper}(A, T)$ be the real spectrum of $\langle A, T \rangle$ and let Y_T^* be the compact Hausdorff subspace of closed points of Y_T (see 4.12).

[2] This is an analog to the well-known result that signatures on a field correspond to (total) orders, and conversely.

a) Each $\alpha \in Y_T$ gives rise to a signature on S, $\tau_\alpha : S \longrightarrow \mathbb{Z}_2$, given by

$$\tau_\alpha(x) \;=\; \begin{cases} 1 & \text{if } x \in \alpha \setminus (-\alpha); \\ -1 & \text{if } x \in -\alpha \setminus \alpha. \end{cases}$$

Clearly, τ_α is a group morphism, taking -1 to -1. For a, $b \in S$, if $b \in \ker\, \tau_\alpha$ and $a = t_1 + t_2 b$, with t_1, $t_2 \in T$ (i.e., $a \in D^v_{S,T}(1,\, b)$), since b, t_1, $t_2 \in \alpha$ and $a \in S \subseteq A^\times$, we get $a \in \alpha \setminus (-\alpha)$, i.e, $\tau_\alpha(a) = 1$, as needed.

b) Note that if $\alpha \subseteq \beta$ in Y_T, then $\tau_\alpha = \tau_\beta$. In particular, if $\alpha \in Y_T$ and $\rho(\alpha)$ is the unique element of Y_T^* extending α (cf. 4.12), then $\tau_\alpha = \tau_{\rho(\alpha)}$. Hence, all signatures induced by orderings on A coincide with those induced by elements of Y_T^*. ∎

If G is a π-SG, then the set of SG-characters of G (cf. 1.12.(b)), is closed in ${\mathbb{Z}_2}^G$ (product topology; discrete topology on $\mathbb{Z}_2$), and so is a Boolean space in the induced topology, called the **space of orders** of G and denoted X_G. Similarly, if S is a T-subgroup of a p-ring $\langle\, A, T\,\rangle$, then $Z_{S,T}$ is a closed subset of ${\mathbb{Z}_2}^S$, and a Boolean space if endowed with the induced topology.

With $S \xrightarrow{q_T} G_T(S)$ the canonical quotient map (cf. 2.5), we have:

Lemma 3.16. *Let S be a T-subgroup of a p-ring $\langle\, A, T\,\rangle$. With notation as above, the map*

$$\sigma \in X_{G_T(S)} \;\longmapsto\; \tau_\sigma = \sigma \circ q_T \in Z_{S,T}$$

is a natural homeomorphism between the space of π-SG-characters of $G_T(S)$ and the space of T-signatures on S. Moreover, for all forms φ over S, $sgn_{\tau_\sigma}(\varphi) = sgn_\sigma(\varphi^T)$.

Proof. Since S, T will remain fixed throughout the proof, write G for $G_T(S)$, X_G for $X_{G_T(S)}$, Z for $Z_{S,T}$ and q for q_T.

Fix $\sigma \in X_G$; we first show that $\tau = \sigma \circ q$ is a T-signature on S. Since q and σ are both group morphisms, taking -1 to -1, the same is true of τ. For a, $b \in S$, assume $a \in D^v_{S,T}(1,\, b)$ and $\tau(b) = 1$. Lemma 2.7.(a.1) yields $q(a) = a^T \in D_G(1,\, q(b)) = D_G(1,\, b^T)$. Since $\sigma(b^T) = \sigma(q(b)) = 1$, and σ is a character (and thus a π-SG morphism), we obtain

$$\sigma(a^T) = \sigma(q(a)) \in D_{\mathbb{Z}_2}(1,\, \sigma(b^T)) \;=\; D_{\mathbb{Z}_2}(1,\, 1),$$

entailing, because $\mathbb{Z}_2$ is a *reduced* special group, $\sigma(a^T) = \tau(a) = 1$, and τ is a signature on S. Since q is onto, we immediately obtain the injectivity of $\sigma \longmapsto \tau_\sigma$. For its surjectivity, fix $\tau \in Z$. Note that $T^\times \subseteq D^v_{S,T}(1,\, 1)$ (even transversally: $u = u/2 + u/2$). Since $1 \in \ker\, \tau$, the definition of signature entails $T^\times \subseteq D^v_{S,T}(1,\, 1) \subseteq \ker\, \tau$. Hence, τ factors uniquely through q, to yield a group morphism, $\sigma : G_T(S) \longrightarrow \mathbb{Z}_2$, such that $\tau = \sigma \circ q = \tau_\sigma$. It remains to check that σ is a morphism of π-SGs. Since $q(-1) = (-1)^T$ and $\tau(-1) = -1$, it is clear that $\sigma(-1) = -1$. Let a, $b \in S$ be such that $a^T \in D_G(1,\, b^T)$; then, 2.7.(a.1) entails $a \in D^v_{S,T}(1,\, b)$; since $D_{\mathbb{Z}_2}(1, -1) = \mathbb{Z}_2$, in order to show that σ is a π-SG morphism it suffices to see that $\sigma(b^T) = 1$ implies $\sigma(a^T) = 1$. If $\sigma(b^T) = \sigma(q(b)) = \tau(b) = 1$, then $b \in \ker\, \tau$ and so $a \in \ker\, \tau$, which in turn yields $1 = \sigma(q(a)) = \sigma(a^T)$, as needed.

Note that for every $\sigma \in X_G$ and $x \in S$, we have

$$\text{(I)} \qquad \sigma(x^T) \;=\; \sigma(q(x)) \;=\; \tau_\sigma(x),$$

which immediately implies that for all forms φ over S, $sgn_\sigma(\varphi^T) = sgn_{\tau_\sigma}(\varphi)$. Regarding continuity, for $a \in S$ and $\delta \in \mathbb{Z}_2$, set

$$\begin{cases} [\![a = \delta]\!]_{X_G} & = \{\sigma \in X_G : \sigma(a^T) = \delta\}; \\ [\![a = \delta]\!]_Z & = \{\tau \in Z : \tau(a) = \delta\}. \end{cases}$$

The collections

$$\begin{cases} \{[\![a = \delta]\!]_{X_G} : a \in S,\, \delta \in \mathbb{Z}_2\}; \\ \{[\![a = \delta]\!]_Z : a \in S \text{ and } \delta \in \mathbb{Z}_2\}, \end{cases}$$

are sub-bases for the Boolean topologies in X_G and Z, respectively. Now, (I) guarantees that for all $a \in S$, $\delta \in \mathbb{Z}_2$ and $\sigma \in X_G$

$$\sigma \in [\![a = \delta]\!]_{X_G} \quad \Leftrightarrow \quad \tau_\sigma \in [\![a = \delta]\!]_Z$$

and the map $\sigma \in X_G \longmapsto \tau_\sigma \in Z$ is indeed a homeomorphism, ending the proof. ■

If $\langle A, T \rangle$ is a p-ring and S is a T-subgroup of A, a natural question is whether all T-signatures on S come from orderings in $\mathrm{Sper}(A,T)$ in the way given by Example 3.15. In general, many distinct orderings in $\mathrm{Sper}(A,T)$ yield the same signature: if $A = \mathbb{R}[X]$ is the polynomial ring in one variable over the real field, the map $\tau : A^\times = \mathbb{R}^\times \longrightarrow \mathbb{Z}_2$, sending positive units in $\mathbb{R}$ to 1 and negative units to -1 is a signature; however, since any ordering on A contains the unique order of $\mathbb{R}$, we have $\tau = \tau_\alpha$ for *every* $\alpha \in \mathrm{Sper}(A)$.

For a T-subgroup of a p-ring $\langle A, T \rangle$ satisfying [T-FQ 2] (in particular, if it is T-faithfully quadratic), we have the following:

LEMMA 3.17. *Let S be a T-subgroup of a p-ring $\langle A, T \rangle$, satisfying* [T-FQ 2]*, let τ be a T-signature on S, and let $n \geq 2$ be an integer.*

a) If $a_1, \ldots, a_n \in \ker \tau$, then $D^v_{S,T}(a_1, \ldots, a_n) \subseteq \ker \tau$.

b) With notation as in 3.15*, there is $\alpha \in Y^*_T$ so that $\tau = \tau_\alpha \restriction S$.*

PROOF. a) If $n = 2$, this follows from the definition of signature on S: indeed, $a \in D^v_{S,T}(a_1,\, a_2)$ entails $a_1 a \in D^v_{S,T}(1,\, a_1 a_2)$ and so $a_1 a \in \ker \tau$; since $\ker \tau$ is a subgroup of S, we obtain $a \in \ker \tau$, as needed. We proceed by induction on $n \geq 2$; if $a_1, \ldots, a_n,\, a_{n+1}$ are in $\ker \tau$, and $a \in D^v_{S,T}(a_1, \ldots, a_n,\, a_{n+1})$ $(a \in S)$, [T-FQ 2] yields $u \in D^v_{S,T}(a_2,\, \ldots,\, a_{n+1})$ such that $a \in D^v_{S,T}(a_1,\, u)$. The induction hypothesis and the case $n = 2$ entail $a \in \ker \tau$, completing the induction step.

b) Let $Q = \{\sum_{i=1}^n a_i t_i : n \in \mathbb{N},\, a_i \in \ker \tau \text{ and } t_i \in T\}$ be the preorder on A generated by $\ker \tau$ and T. We claim that Q is proper; otherwise, $-1 \in Q$, and then there would be $a_1, \ldots, a_n \in \ker \tau$ and $t_1, \ldots, t_n \in T$ such that $-1 = \sum_{i=1}^n a_i t_i$, i.e., $-1 \in D^v_{S,T}(a_1, \ldots, a_n)$, which is impossible by item (a). Let $\beta \in \mathrm{Sper}(A,Q) \subseteq \mathrm{Sper}(A,T)$ (which exists by [**BCR**], Propositions 4.3.8, p. 90 and, 4.2.7, p. 87). With notation as in 3.15.(b), let α the unique extension of β in Y^*_T. Since τ is a T-signature on S, $\ker \tau$ $(\subseteq S \cap \alpha \subseteq A^\times \cap \alpha)$ is a subgroup of index 2 in S; since $\ker \tau \subseteq \ker \tau_\beta \restriction S$ (by construction), we conclude that for all $a \in S$, $\tau(a) = 1 \Leftrightarrow a \in \ker \tau \Leftrightarrow \tau_\alpha(a) = 1$, and so $\tau = \tau_\alpha \restriction S$, as claimed. ■

We now state

PROPOSITION 3.18. (Pfister's local-global principle for T-isometry) *Let $\langle A, T \rangle$ be a p-ring and let S be a T-subgroup of A. Let $Y_T = \mathrm{Sper}(A,T)$, let Y^*_T be the* *subspace of closed points in Y_T and let K be a dense subset of Y_T^* in the spectral topology. If S is T-faithfully quadratic, then for all forms φ, ψ of the same dimension over S, the following are equivalent:*

(1) $\varphi \approx_T^S \psi$; (2) *For all* $\tau \in Z_{S,T}$, $sgn_\tau(\varphi) = sgn_\tau(\psi)$.

(3) *For all* $\alpha \in Y_T^*$, $sgn_{\tau_\alpha}(\varphi) = sgn_{\tau_\alpha}(\psi)$.

(4) *For all* $\beta \in K$, $sgn_{\tau_\beta}(\varphi) = sgn_{\tau_\beta}(\psi)$.

PROOF. It is clear that (3) $\Rightarrow$ (4), while (2) $\Leftrightarrow$ (3) follows immediately from Example 3.15 and Lemma 3.17.(b). It remains to prove that (1) $\Leftrightarrow$ (2) and (4) $\Rightarrow$ (3).

(1) $\Leftrightarrow$ (2) : We recall Proposition 3.7, p. 51 in [**DM2**]:

(I) *If G is a reduced special group, then for all forms φ, ψ of the same dimension over G,* $\varphi \equiv_G \psi \;\Leftrightarrow\;$ *For all* $\sigma \in X_G$, $sgn_\sigma(\varphi) = sgn_\sigma(\psi)$.

Let φ, ψ be forms of the same dimension over S. Since S is T-faithfully quadratic, by Theorems 3.5 and 3.6, we have,

$$\text{(II)} \qquad \varphi \approx_T^S \psi \quad \Leftrightarrow \quad \varphi^T \equiv_T^S \psi^T.$$

Recalling that $\equiv_T^S$ is isometry in the *reduced special group* $G_T(S)$, (I) and Lemma 3.16 entail

$$\text{(III)} \qquad \begin{aligned} \varphi^T \equiv_T^S \psi^T \quad &\Leftrightarrow \quad \text{For all } \sigma \in X_{G_T(S)},\ \ sgn_\sigma(\varphi^T) = sgn_\sigma(\psi^T) \\ &\Leftrightarrow \quad \text{For all } \tau \in Z_{S,T},\ \ sgn_\tau(\varphi) = sgn_\tau(\psi). \end{aligned}$$

From (II) and (III) we immediately conclude the equivalence of (1) and (2), as desired.

(4) $\Rightarrow$ (3) : If φ, ψ are forms of the same dimension over S, set

$$C \;=\; \{\alpha \in Y_T^* : sgn_{\tau_\alpha}(\varphi) = sgn_{\tau_\alpha}(\psi)\}.$$

It is straightforward that C is a closed set in Y_T^*, containing K, whence we obtain $C = Y_T^*$, as needed. ∎

REMARK 3.19. Let S be a T-subgroup of a p-ring $\langle A, T \rangle$. With notation as in Proposition 3.18, it is clear that the condition

$$(*) \qquad \text{For all } \alpha \in \mathrm{Sper}(A,T),\ \ sgn_{\tau_\alpha}(\varphi) = sgn_{\tau_\alpha}(\psi)$$

entails (3) in 3.18; in fact, by 3.15.(b), they are equivalent. However, in concrete examples, conditions (3) and (4) in 3.18 may facilitate the proof of certain properties of forms over S (see Theorem 10.7 and Remark 10.9). ∎

CHAPTER 4

Idempotents, Products and T-isometry

This chapter collects basic results concerning orthogonal decompositions of a ring into idempotents, as well as the relation between T-isometry and products, that shall be fundamental in what follows. The notational conventions and results in 2.27 may be used without further comment.

1. T-isometry under Orthogonal Decompositions and Products

4.1. **The Boolean Algebra of Idempotents of a Ring.**

a) If A is a ring, write $B(A)$ for the Boolean algebra of idempotents in A (for more details, see 5.1 in [**DM8**]) . Recall that for $e, f \in B(A)$

(*i*) $e \wedge f := ef$; (*ii*) $e \vee f := e + f - ef$;

(*iii*) $e \bigtriangleup f = e + f - 2ef$,

where $\bigtriangleup$ is symmetric difference in $B(A)$. Note that the top and bottom of $B(A)$ are 1 and 0, respectively; and the *complement* of e in $B(A)$ is $1 - e$. Moreover, $e, f \in B(A)$ are said to be *orthogonal or disjoint* if $ef = 0$. If $e \in B(A)$, then Ae is a ring with identity e.

b) The following is straightforward:

Fact 4.2. *Let A be a ring and $e \neq 0$ be an idempotent of A. Then,*

1) *Ae is a ring with e as its multiplicative identity and 2 is invertible in Ae, i.e., $2e \in (Ae)^{\times}$.*

2) *If $T = A^2$ or T is a preorder of A, then Te is $(Ae)^2$ or a preorder of Ae.*

3) *The map $\rho_e : A \longrightarrow Ae$ given by $\rho_e(a) = ae$ is a p-ring morphism from $\langle A, T \rangle$ onto $\langle Ae, Te \rangle$.*

4) *If S is a T-subgroup of A, then Se is Te-subgroup of Ae.* ■

c) An **orthogonal decomposition** of A is a sequence $e_1, \ldots, e_m$, of non-zero, pairwise disjoint idempotents in A, such that $\sum_{k=1}^{m} e_k = 1$.

An orthogonal decomposition $\{e_1, \ldots, e_m\}$ of A induces a ring isomorphism between the product $\prod_{k=1}^{m} Ae_k$ and A, given by $\langle a_1, \ldots, a_m \rangle \longmapsto \sum_{k=1}^{m} a_k$. ■

The proof of the following result is straightforward and will be omitted:

Lemma 4.3. *Let A be a ring and let $e_1, \ldots, e_m$ be an orthogonal decomposition of A.*

a) For $a_1, \ldots, a_m \in A$, let $a = \sum_{k=1}^{m} a_k e_k$. Then,

(1) *$a \in A^{\times}$ ⇔ For all $1 \leq k \leq m$, $a_k e_k \in (Ae_k)^{\times}$.*

(2) *If T is a proper preorder of A,*

$$a \in T^{\times} \;\Leftrightarrow\; \text{For all } 1 \leq k \leq m, \;\; a_k e_k \in (Te_k)^{\times}.$$

In particular, these equivalences hold if a_k is in $(Ae_k)^{\times}$ (resp., $(Te_k)^{\times}$), for all $1 \leq k \leq n$.

b) For $1 \leq k \leq m$, let $\varphi_k = \langle a_{k1}, \ldots, a_{kn} \rangle$ be a non-singular diagonal quadratic form over Ae_k. For $1 \leq j \leq n$, set $a_j = \sum_{k=1}^{m} a_{kj}$. Then, $\varphi = \langle a_1, \ldots, a_n \rangle$ is a non-singular diagonal quadratic form over A, such that $d(\varphi) = \sum_{k=1}^{m} d(\varphi_k)$, where d denotes the discriminant. ■

To deal with products, we set down some notation and register a set of elementary properties of the concepts involved.

4.4. **Notation and Remarks.** a) Let R be a ring and let $P = R^2$ or a preorder of R. If $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_n \rangle$ are n-forms over R, write $\varphi \cdot \psi \in P^{\times}$ to stand for $a_j b_j \in P^{\times}$, $1 \leq j \leq n$.

We then clearly have

$$\text{(I)} \qquad \left\{ \begin{array}{c} \exists\, t_1, \ldots, t_n \in P^{\times} \text{ such that for all } 1 \leq i \leq n, \;\; b_i = t_i a_i \\ \Updownarrow \\ \varphi \cdot \psi \in P^{\times}. \end{array} \right.$$

It follows from (I) that condition (ii) in the definition of P-isometry (2.17) can be rewritten as

$(ii)'$ For all $1 \leq i \leq k$, $\varphi_i \approx \varphi_{i-1}$ or $\varphi_i \cdot \varphi_{i-1} \in P^{\times}$.

We shall use this equivalence without further comment in what follows.

b) Let I be a non-empty set, let A_i, $i \in I$, a family of rings and let $A = \prod_{i \in I} A_i$ be their product. Write $p_i : A \longrightarrow A_i$ for the canonical projection. Notation is as in 2.27.(b)

(1) If $M_i = (a_i^{jk})$, $i \in I$, are $n \times n$ matrices with entries in A_i, write $\prod_{i \in I} M_i$ for the $n \times n$ matrix in A whose $(jk)^{th}$-entry is the I-sequence $\langle a_i^{jk} \rangle_{i \in I}$.

(2) If $M = (a^{jk})$ and $N = (b^{jk})$ are $n \times n$ matrices with coefficients in A, then

$$\begin{aligned} M = N \;&\Leftrightarrow\; \text{For all } 1 \leq j,\, k \leq n, \;\; a^{jk} = b^{jk} \\ &\Leftrightarrow\; \text{For all } 1 \leq j,\, k \leq n \text{ and for all } i \in I, \;\; p_i(a^{jk}) = p_i(b^{jk}) \\ &\Leftrightarrow\; \text{For all } i \in I, \;\; p_i(M) = p_i(N). \end{aligned}$$

From this and item (b) in 2.27 we obtain the following relations, where M_i, N_i are $n \times n$ matrices with entries in A_i, $i \in I$:

2.1 If M is a $n \times n$ matrix in A, $M = \prod_{i \in I} p_i(M)$.

2.2 $\left(\prod_{i \in I} M_i\right) \left(\prod_{i \in I} N_i\right) = \prod_{i \in I} M_i N_i$.

2.3 $\left(\prod_{i \in I} M_i\right)^t = \prod_{i \in I} M_i^t$.

2.4 $\det \left(\prod_{i \in I} M_i\right) = \langle \det M_i \rangle_{i \in I} \in A$.

2.5 $\prod_{i \in I} M_i \in \mathrm{GL}_n(A)$ iff $M_i \in \mathrm{GL}_n(A_i)$, for all $i \in I$.

c) Let $n \geq 1$ be an integer and for $i \in I$, let $\varphi_i = \langle a_{1i}, \ldots, a_{ni} \rangle$ be a n-form over A_i. Write $\prod_{i \in I} \varphi_i$ for the n-form over A, whose j^{th}-entry is the I-sequence $\langle a_{ji} \rangle_{i \in I}$,

$1 \leq j \leq n$. Note that,

(1) For all $j \in I$, $p_j \star (\prod_{i\in I} \varphi_i) = \varphi_j$.

(2) For any form φ over A, $\varphi = \prod_{i\in I} p_i \star \varphi$.

(3) With notation as in 2.4.(2), $\mathcal{M}(\prod_{i\in I} \varphi_i) = \prod_{i\in I} \mathcal{M}(\varphi_i)$.

d) For $i \in I$, write Id_n^i for the identity matrix in $\mathrm{GL}_n(A_i)$. ■

PROPOSITION 4.5. *Let A_i, $i \in I$, be a non-empty family of rings. For each $i \in I$, Let $T_i \subseteq A_i$. Assume that for all $i \in I$, T_i is a preorder of A_i, or for all $i \in I$, $T_i = A_i^2$, and set $T = \prod_{i\in I} T_i$* [1]. *Let S_i be a T_i-subgroup of A_i and let $S = \prod_{i\in I} S_i$ be the corresponding T-subgroup of $A = \prod_{i\in I} A_i$. For each $i \in I$, let φ_i, ψ_i be n-forms over S_i. Consider the following conditions:*

(1) $\prod_{i\in I} \varphi_i \approx_T^S \prod_{i\in I} \psi_i$.

(2) *For all $i \in I$,* $\varphi_i \approx_{T_i}^{S_i} \psi_i$.

We have:

a) (1) $\Rightarrow$ (2).

b) For each $i \in I$, let $\ell(i)$ be the length of a witnessing sequence for $\varphi_i \approx_{T_i}^{S_i} \psi_i$, and suppose the following condition is satisfied

$$(*) \qquad \ell := \sup \{\ell(i) : i \in I\} \text{ is finite.}$$

Then, (2) $\Rightarrow$ (1).

c) If I is finite, or if each S_i is T_i-faithfully quadratic, then condition $(*)$ in (b) holds for all pairs of T_i-isometric forms over S_i, of any dimension $n \geq 1$. In particular, in these cases, (1) *and* (2) *are equivalent.*

d) Let R be a ring and P be a preorder of R or $P = R^2$. Let $\{e_1, \ldots, e_m\}$ be an orthogonal decomposition of R into idempotents and let $\varphi_k = \langle a_{k1}, \ldots, a_{kn} \rangle$, $\psi_k = \langle b_{k1}, \ldots, b_{kn} \rangle$ be non-singular diagonal quadratic forms over Re_k $(1 \leq k \leq m)$. Set $a_j = \Sigma_{k=1}^m a_{kj}$, $b_j = \Sigma_{k=1}^m b_{kj}$, for $1 \leq j \leq n$, and $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_n \rangle$. Then,

$$\varphi \approx_P \psi \quad \Leftrightarrow \quad \text{For all } 1 \leq k \leq m, \;\; \varphi_k \approx_{Pe_k} \psi_k.$$

PROOF. Item (a) is a straightforward consequence of 2.27.(d).

b) We shall freely use the relations established in 4.4. Write $\varphi = \prod_{i\in I}\varphi_i$ and $\psi = \prod_{i\in I} \psi_i$.

Since ℓ is finite, we may assume, without loss of generality, that for all $i \in I$, the T_i-isometry $\varphi_i \approx_{T_i}^{S_i} \psi_i$ has a witnessing sequence of length ℓ (by repeating, if necessary, the last form of a given sequence connecting φ_i to ψ_i), $\varphi_i = \theta_0^i, \ldots, \theta_p^i, \ldots, \theta_\ell^i = \psi_i$. It suffices to give a uniform method to show that the passage from the p^{th}-form $(0 \leq p \leq \ell - 1)$ to $(p+1)^{th}$-form in each coordinate, produces a witnessing sequence for the T-isometry between φ and ψ in finitely many steps. To simplify notation, we write, for each $i \in I$, α_i for θ_p^i and β_i for θ_{p+1}^i, setting $\alpha = \prod_{i\in I} \alpha_i$ and $\beta = \prod_{i\in I} \beta_i$. Define

[1] In general, if A, B are rings, T is a preorder of A and $T' = B^2$, then $T \times T'$ is neither a preorder of $A \times B$, nor $(A \times B)^2$. Consider, e.g., $A = B = \mathbb{Q}$, $T = \mathbb{Q}^+$ and $T' = \mathbb{Q}^2$.

$$\begin{cases} J &= \{j \in I : \alpha_i \cdot \beta_i \in T_i^{\times}\}; \\ K &= \{k \in I : \exists\, M_k \in \mathrm{GL}_n(A_k) \text{ with } \mathcal{M}(\beta_k) = M_k \mathcal{M}(\alpha_k) M_k^t\}. \end{cases}$$

Clearly, $I = J \cup K$; moreover, we may assume J and K are disjoint (otherwise, employ $J \cup (K \setminus J)$). For each $k \in K$, fix $M_k \in \mathrm{GL}_n(A_k)$ with the required property. There are three possibilities to discuss:

(i) $J = \emptyset$: Then $K = I$, and we obtain

$$\begin{aligned}
&(\textstyle\prod_{k\in K} M_k)\ \mathcal{M}(\alpha)\ (\prod_{k\in K} M_k)^t \ = \\
&\quad = (\textstyle\prod_{k\in K} M_k)\ (\prod_{k\in K} \mathcal{M}(\alpha_k))\ (\prod_{k\in K} M_k)^t \ = \\
&\quad = (\textstyle\prod_{k\in K} M_k)\ (\prod_{k\in K} \mathcal{M}(\alpha_k))\ (\prod_{k\in K} M_k^t) \ = \\
&\quad = \textstyle\prod_{k\in K} M_k \mathcal{M}(\alpha_k) M_k^t \ = \ \prod_{k\in K} \mathcal{M}(\beta_k) \ = \ \mathcal{M}(\beta),
\end{aligned}$$

as needed.

(ii) $K = \emptyset$: Here $J = I$, and $\alpha \cdot \beta = \prod_{j\in I} \alpha_j \cdot \beta_j \ \in \ T^{\times}$, as needed.

(iii) $J \neq \emptyset$ and $K \neq \emptyset$: We claim that the sequence

$$\alpha, \quad \gamma := \textstyle\prod_{j\in J} \beta_j \times \prod_{k\in K} \alpha_k, \quad \beta$$

witnesses the T-isometry of α and β. Indeed, we have:

- $\alpha \cdot \gamma \ = \ \prod_{j\in J} \alpha_j \cdot \beta_j \times \prod_{k\in K} \alpha_k \cdot \alpha_k \ \in \ T^{\times}$;
- If $M = \prod_{j\in J} Id_n^j \times \prod_{k\in K} M_k$, a computation similar to that in case (i) shows that $M\mathcal{M}(\gamma)M^t = \mathcal{M}(\beta)$, completing the proof of (b).

c) The statement is clear if I is finite, while for the case in which all S_i are T_i-faithfully quadratic it follows from Proposition 5.1 below.

Item (d) is an immediate consequence of (a), (b) and the fact that $\langle R, P \rangle$ is p-ring isomorphic to $\prod_{k=1}^m \langle Re_k, Pe_k \rangle$. ■

Our next order of business is to establish the following

THEOREM 4.6. *With the same notation and conventions as in the statement of* 4.5*, the following are equivalent:*

(1) *For all $i \in I$, S_i is T_i-faithfully quadratic;*

(2) *S is T-faithfully quadratic.*

In particular, the theory of T-faithfully quadratic rings is preserved by arbitrary products.

PROOF. The last assertion is clearly a consequence of the stated equivalence. Notation is as in 4.4.

(1) ⇒ (2) : Since axioms [T-FQ 1] and [T-FQ 2] are described by Horn-geometric sentences (cf. proof of 5.2), they are preserved by products (cf. [**CK**], Prop. 6.6.2). For [T-FQ 3], let $\langle \overline{a} \rangle \oplus \varphi \approx_T^S \langle \overline{a} \rangle \oplus \psi$, with $\overline{a} = \langle a_i \rangle_{i\in I} \in S$ and φ, ψ forms of the same dimension over S. Hence, for all $i \in I$, $\langle a_i \rangle \oplus (p_i \star \varphi) \ \approx_{T_i}^{S_i} \langle a_i \rangle \oplus (p_i \star \psi)$; since [$T_i$-FQ 3] holds in $\langle S_i, T_i \rangle$, we get $(p_i \star \varphi) \approx_{T_i}^{S_i} (p_i \star \psi)$, and 4.5.(c) yields $\varphi \approx_T^S \psi$, as needed.

(2) ⇒ (1): Fix $i \in I$; we prove that S_i is T_i-faithfully quadratic.

We define two maps from $A_i \longrightarrow A = \prod_{i \in I} A_i$, as follows:

(A) Let $\cdot^* : A_i \longrightarrow A$ be the map $a \longmapsto a^*$, where

$$a^*(j) \;=\; \begin{cases} a & \text{if } j = i; \\ 1 & \text{if } j \neq i. \end{cases}$$

Clearly, this map preserves product and takes $1 \in A_i$ to $1 \in A$. Moreover,

(I) $$\begin{cases} a \in A_i^{\times} \Leftrightarrow a^* \in A^{\times}, \quad a \in S_i \Leftrightarrow a^* \in S \quad \text{and} \\ a \in T_i \quad \Leftrightarrow \quad a^* \in T. \end{cases}$$

If $\varphi = \langle\, a_1, \ldots, a_n \,\rangle$ is a n-form over S_i, write φ^* for the n-form over S given by $\langle\, a_1^*, \ldots, a_n^* \,\rangle$. Note that $p_i \star \varphi^* = \varphi$ and $p_j \star \varphi^* = n\langle\, 1 \,\rangle$, for all $j \neq i$. Hence, 4.5.(b) immediately entails

(II) For all n-forms over S_i, $\varphi \approx^{S_i}_{T_i} \psi \;\Leftrightarrow\; \varphi^* \approx^{S}_{T} \psi^*$.

(B) Let $\cdot_* : A_i \longrightarrow A$ be the map $a \longmapsto a_*$, where for $j \in I$,

$$a_*(j) \;=\; \begin{cases} a & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

This map preserves product, sum and takes $0 \in A_i$ to $0 \in A$. Moreover, $t \in T_i$ iff $t_* \in T$.

We now observe that for $a \in S_i$ and a n-form φ over S_i ,

(III) $a \in D^v_{S_i,T_i}(\varphi) \;\Leftrightarrow\; a^* \in D^v_{S,T}(\varphi^*)$.

Indeed, the implication ($\Leftarrow$) in (III) follows from the fact that the projection p_i preserves value representation (cf. 2.27.(d)). For the converse, we have:

• If T is a preorder of A, then $a \in D^v_{S_i,T_i}(\varphi)$ entails $a = \sum_{k=1}^n t_k b_k$, with $t_k \in T_i$. It is straightforward that

$$a^* \;=\; t_1^*\, b_1^* + (t_2)_*\, b_2^* + \cdots + (t_n)_*\, b_n^*,$$

with t_1^*, $(t_i)_*$ $(i \geq 2)$ in T, yielding the desired conclusion.

• If $T = A^2$, then $a \in D^v_{S_i,T_i}(\varphi)$ entails $a = \sum_{k=1}^n x_k^2 b_k$, with $x_k \in A_i$, $1 \leq k \leq n$. Then,

$$a^* \;=\; (x_1^*)^2\, b_1^* + ((x_2)_*)^2\, b_2^* + \cdots + ((x_n)_*)^2\, b_n^*,$$

whence, $a^* \in D^v_{S,T}(\varphi^*)$, establishing (III).

We now complete the proof of Theorem 4.6.

<u>S_i verifies [(T_i)-FQ 1] :</u> Suppose $a \in D^v_{S_i,T_i}(b,\, c)$, with $a,\, b,\, c \in S_i$. By (III), $a^* \in D^v_{S,T}(b^*, c^*)$. Since [T-FQ 1] holds in A, we obtain $a^* \in D^t_{S,T}(b^*,\, c^*)$ and the fact that p_i is a p-ring morphism and 2.27.(d) entail $a \in D^t_{S_i,T_i}(b,\, c)$.

<u>S_i verifies [(T_i)-FQ 2] :</u> Let $\varphi = \langle\, b_1, \ldots, b_n \,\rangle$ be a form over S_i; by 2.26.(c), it suffices to show that $D^v_{S_i,T_i}(\varphi) \subseteq \mathfrak{D}_{S_i,T_i}(\varphi)$. Let $a \in S_i$ be such that $a \in D^v_{S_i,T_i}(\varphi)$ and fix $1 \leq k \leq n$. By (III), $a^* \in D^v_{S,T}(\varphi^*)$; since S satisfies [T-FQ 2], there is $u \in S$ such that $u \in D^v_{S,T}(b_1^*, \ldots, \check{b}_k^*, \ldots, b_n^*)$ and $a^* \in D^v_{S,T}(b_k^*,\, u)$. But then, $p_i(u) \in S_i$, and 2.27.(d) yields $a \in D^v_{S_i,T_i}(b_k, p_i(u))$ and $p_i(u) \in D^v_{S_i,T_i}(b_1, \ldots, \check{b}_k, \ldots, b_n)$, as needed.

<u>S_i verifies [(T_i)-FQ 3]</u> : Let $a \in S_i$ and let φ, ψ be n-forms over S_i ($n \geq 1$) such that $\langle a \rangle \oplus \varphi \approx^{S_i}_{T_i} \langle a \rangle \oplus \psi$. But then, (II) implies $\langle a^* \rangle \oplus \varphi^* \approx^{S}_{T} \langle a^* \rangle \oplus \psi^*$. Since A satisfies [T-FQ 3], we obtain $\varphi^* \approx^{S}_{T} \psi^*$, and another application of (II) yields $\varphi \approx^{S_i}_{T_i} \psi$, completing the proof. ■

Theorem 4.6 yields

COROLLARY 4.7. *Let A be a ring and let T be A^2 or a proper preorder of A. Let S be a T-faithfully quadratic T-subgroup of A. If $e \neq 0$ is an idempotent in A, then Se is Te-faithfully quadratic.* ■

A proper preorder on a product may project to the trivial p-ring, (A, A), on some coordinate (an example of this situation occurs in the proof of Fact 2, Theorem 8.21). In this respect, we register:

LEMMA 4.8. *The trivial p-ring, $\langle A, A \rangle$ is A-faithfully quadratic and the associated special group is the trivial SG, i.e., $G_A(A) = \{1\}$. In particular, all forms over $A^\times$ are transversally universal and any two forms over $A^\times$ of the same dimension are A-isometric.*

PROOF. Let $\varphi = \langle x_1, \ldots, x_n \rangle$ and $\psi = \langle y_1, \ldots, y_n \rangle$ be forms over $A^\times$. Then, since $T^\times = A^\times$, $\varphi = \langle y_1(x_1/y_1), \ldots, y_n(x_n/y_n) \rangle$, showing that all forms of the same dimension are A-isometric. The fact that all forms are transversally universal is an immediate consequence of the standing assumption $2 \in A^\times$ and the following

FACT 4.9. *For all $n \geq 2$, there are integers $k_1, \ldots, k_n \geq 1$ such that $1 = \sum_{j=1}^n 1/2^{k_j}$.* ■

For c, $a_1, \ldots, a_n \in A^\times$, if $1 = \sum_{j=1}^n \varepsilon_j$ is a decomposition of 1 as in 4.9, we obtain $c = \sum_{j=1}^n c(\varepsilon_j/a_j)\, a_j$, with $c(\varepsilon_j/a_j) \in A^\times$, $1 \leq j \leq n$, as needed to show that $c \in D^t_{A,A}(a_1, \ldots, a_n)$. These observations immediately imply that $\langle A, A \rangle$ verifies axioms [A-FQ 1], [A-FQ 2] and [A-FQ 3]; for [A-FQ 2], note that for any $1 \leq j \neq k \leq n$, we have $c \in D^t_{A,A}(a_j, a_k)$, with $a_k \in A^\times$ represented by $\langle a_1, \ldots, \check{a}_j, \ldots, a_n \rangle$. Clearly, $G_A(A) = A^\times/A^\times = \{1\}$, as stated. ■

2. The Real Spectrum. Summary of Basic Properties

In later chapters we shall use some basic algebraic and topological properties of the real spectrum of a ring, as well as its relationship with that of its localization at an idempotent. In this section we collect, omitting proofs, the necessary results on these matters, mostly belonging to its "folklore".

For the definition and basic properties of the real spectrum of a p-ring $\langle A, T \rangle$, written $\mathrm{Sper}(A, T)$, the reader is referred to any one of the survey papers [**Be1**], [**Di**], [**K**] or to Chapter 7 of the book [**BCR**].

4.10. **Algebraic Properties.** a) Fix a preorder T of A. Every element $a \in A$ gives rise to a map, $\overline{a}_T : \mathrm{Sper}(A,T) \longrightarrow \mathbf{3} = \{-1,0,1\}$, defined by: for $\alpha \in \mathrm{Sper}(A,T)$,

$$\overline{a}_T(\alpha) = \begin{cases} 1 & \text{if } a \in \alpha \setminus (-\alpha); \\ 0 & \text{if } a \in \mathrm{supp}(\alpha) = \alpha \cap -\alpha; \\ -1 & \text{if } a \in (-\alpha) \setminus \alpha. \end{cases}$$

Each $\alpha \in \mathrm{Sper}(A,T)$ gives rise to linearly ordered domain $\langle A_\alpha, \leq_\alpha \rangle$, where $A_\alpha = A/\mathrm{supp}(\alpha)$ and $x/\alpha := x/\mathrm{supp}(\alpha) \geq_\alpha 0$ iff $x \in \alpha$. Since $a \in \alpha \setminus (-\alpha)$ iff $\pi_\alpha(a) >_\alpha 0$, $\overline{a}_T(\alpha)$ is just the sign of $\pi_\alpha(a)$ in the linearly ordered domain A_α. Whenever T is clear from context its mention will be omitted from the notation, and we write $\overline{a}$ for $\overline{a}_T$. We also write 0, 1 and -1 for the maps $\overline{0}$, $\overline{1}$ and $\overline{-1}$.

b) Some of the fundamental properties of the maps $\overline{a}$ can be found in section 4 of Chapter 5 in [**Mar**]. We register here the ones that are most frequently used in the present text. For items (a), (b) and (c) of the next result, see Theorem 5.4.2 (p. 93) in [**Mar**]; for (d), see Corollary 5.4.3 (p. 94) in [**Mar**].

THEOREM 4.11. *Let T be a preordering on A and let a, $b \in A$.*

a) $\overline{a} = 0$ *iff there is $k \geq 0$ such that $-a^{2k} \in T$.*

b) $\overline{a} = 1$ *iff there are s, $t \in T$ such that $(1+s)a = 1+t$.*

c) $\overline{a} \geq 0$ *iff there are s, $t \in T$ and $k \geq 0$ such that $(a^{2k} + s)a = a^{2k} + t$.*

d) $\overline{a} = \overline{b}$ *iff there are s, $t \in T$ and $k \geq 0$ such that $sab = (a^2 + b^2)^k + t$.* ■

4.12. **Topological Properties.** As is well-known, endowed with its spectral topology (cf. [**BCR**], Def. 7.1.3, p. 133) the set $\mathrm{Sper}(A,T)$ is a normal spectral space. Even more: the set of specializations of any point is totally ordered under specialization ([**BCR**], Prop. 7.1.23, p. 141).[2]

As usual, if Z is a topological space and $K \subseteq Z$, $\overline{K}$ denotes the closure of K in Z, and $B(Z)$ the Boolean algebra of clopens in Z.

Let Y be a normal spectral space and let Y^* be its subspace of closed points. Let $\iota : Y^* \longrightarrow Y$ be the inclusion, clearly a continuous map. In Proposition 2 of [**CC**] (p. 230; see also Propositions 7.1.23 and 7.1.24 in [**BCR**]) it is shown that normality of Y is equivalent to

(*) For all $y \in Y$, $\overline{\{y\}} \cap Y^*$ is a singleton, written $\{\rho(y)\}$,

yielding a map $\rho : Y \longrightarrow Y^*$, such that $\rho \circ \iota = Id_{Y^*}$. In the same reference it is shown that

(1) With the induced topology, Y^* is a compact Hausdorff space;

(2) The map ρ is a closed continuous retraction, with section ι.

Recall that if $f : D \longrightarrow E$ is a continuous map of topological spaces, the **Stone dual** of f is the map $f^* : B(E) \longrightarrow B(D)$, given by $f^*(V) = f^{-1}[V]$. The function f^* is a morphism of Boolean algebras and the pair of maps $\langle E \mapsto B(E),\ f \mapsto f^* \rangle$ is a (contravariant) functor from the category of topological spaces to the category of Boolean algebras. ■

[2] Additional references for this topic are [**CC**] and sections 6.3, 6.4 in [**Mar**].

We also note

LEMMA 4.13. *With notation as in* 4.12,

a) *If* Y *is a normal spectral space and* Y^* *is its subspace of closed points, then* ι^* *and* ρ^* *are inverse Boolean algebra isomorphisms between* $B(Y^*)$ *and* $B(Y)$.

b) *If* $\langle A, T \rangle$ *is a p-ring,* $Y_T = \mathrm{Sper}(A, T)$ *is a normal spectral space. Write* Y_T^* *for the subspace of closed points in* Y_T, *and* ι_T, ρ_T *for the associated continuous maps* (*as in* 4.12). *Then,*

(1) ρ_T^* *and* ι_T^* *are inverse Boolean algebra isomorphisms between* $B(Y_T^*)$ *and* $B(Y_T)$.

(2) *For all* $a \in A^\times$ *and* $\varepsilon \in \{\pm 1\}$,

$$\rho_T^{-1}[[\![\overline{a} = \varepsilon]\!] \cap Y_T^*] \quad = \quad \rho_T^*([\![\overline{a} = \varepsilon]\!] \cap Y_T^*) \quad = \quad [\![\overline{a} = \varepsilon]\!].$$

c) *Let* Z *be a completely regular topological space and let* $A = \mathbb{C}(Z)$ *be the* $\mathbb{R}$*-algebra of real-valued continuous functions on* Z. *If* $Y = \mathrm{Sper}(A)$, *the map* $\eta : Z \longrightarrow Y^*$, *given by*

$$\eta(z) \quad = \quad \alpha_z \quad = \quad \{f \in A : f(z) \geq 0\}$$

is a homeomorphism from Z *onto a dense subset of* Y^*; *in particular,* $B(Z)$ *and* $B(Y^*)$ *are isomorphic.* ■

4.14. **The Real Spectrum under Orthogonal Decompositions.** For ease of reading we recall the following notions and results.

a) An ideal I in a p-ring $\langle A, T \rangle$ is

- **T-convex** if for all $s, t \in T$, $s + t \in I \Rightarrow s, t \in I$;
- **T-radical** if for all $a \in A$ and $t \in T$, $a^2 + t \in I \Rightarrow a \in I$.

A ΣA^2-radical ideal is called **real**.

By Proposition 4.2.5 in [**BCR**] an ideal of A is T-radical iff it is T-convex and radical. In particular, a prime ideal is T-radical iff it is T-convex.

Note that if $T \subseteq \alpha \in \mathrm{Sper}(A)$, then the prime ideal $\mathrm{supp}(\alpha) := \alpha \cap -\alpha$ is T-convex. Conversely,

PROPOSITION 4.14.1 ([**BCR**], Prop. 4.3.8, p. 90) *If* I *is a proper prime ideal that is* T*-convex for a given preorder* T *of* A, *then there is* $\alpha \in \mathrm{Sper}(A, T)$ *such that* $\mathrm{supp}(\alpha) = I$. ■

PROPOSITION 4.14.2 ([**BCR**], Prop. 4.2.7, p. 87)

(1) *A preorder* T *on a ring* A *is proper iff* A *has a proper* T*-convex ideal.*

(2) *If* T *is a preorder of* A, *any ideal of* A, *maximal for the property of being* T*-convex, is prime.* ■

PROPOSITION 4.14.3 ([**BCR**], Prop. 4.2.6, p. 87) *Given a preorder* T *of* A, *every ideal* I *of* A *is contained in a smallest* T*-radical ideal* (*possibly improper*), *namely:*

$$\sqrt[T]{I} \quad = \quad \{a \in A : \exists\, m \in \mathbb{N} \text{ and } t \in T \text{ such that } a^{2m} + t \in I\},$$

*called the **T-radical** of I, the intersection of all T-convex prime ideals containing I.* ■

b) With notation as above:

(b.1) If $a \in A$, write $\sqrt[T]{a}$ for the T-radical of the principal ideal (a). By 4.14.3, an ideal I is T-radical iff $\sqrt[T]{I} = I$.

(b.2) If $T = \Sigma A^2$ and I is an ideal in A, write $\sqrt[\mathrm{re}]{I}$ for $\sqrt[T]{I}$, the **real radical** of I, i.e., the intersection of all real primes of A containing I.

(b.3) Recall that a ring A is **reduced** if it has no non-zero nilpotent elements or, equivalently, the intersection of all prime ideals in A is the zero ideal, (0).

c) The next definition describes the analog of the notion of reduced in the case of preordered rings and that of a completely real ring.

DEFINITION 4.14.4 *a) A p-ring $\langle A, T \rangle$ is **T-reduced** if $\sqrt[T]{0} = (0)$. In case $T = \Sigma A^2$, i.e., $\sqrt[re]{0} = (0)$, we say that A is a **real ring**.*

b) (Definition 2.1.6 and Proposition 2.1.7, p. 18, [**Be2**]) A ring A is **completely real** if it is reduced (4.14.(b.3)) and every prime ideal of A is real. ■

Clearly, any T-reduced ring is reduced and semi-real, and any completely real ring is real. Here are the basic properties of completely real rings:

LEMMA 4.15. *If A is a completely real ring, then*

a) $\sqrt[re]{0} = (0)$ and A has weak bounded inversion, i.e., $1 + \Sigma A^2 \subseteq A^{\times}$. In particular, A has a natural structure of $\mathbb{Q}$-algebra.

b) The map $\mathrm{supp} : \mathrm{Sper}(A) \longrightarrow \mathrm{Spec}(A)$, is a continuous surjection, where $\mathrm{Spec}(A)$ is the Zariski spectrum of A.

PROOF. Since every prime ideal in A is real and A is reduced, we must have $\sqrt[\mathrm{re}]{0} = (0)$; in particular, every maximal ideal is real and so 7.2 entails that A has weak bounded inversion. If $n \geq 1$ is an integer, then $n \in 1 + \Sigma A^2 \subseteq A^{\times}$ and A is a $\mathbb{Q}$-algebra, establishing (a). Item (b) is Proposition 7.1.8, p. 136, in [**BCR**]. ■

The main results relating the real spectrum of a preordered ring to that of its localization at an idempotent are contained in the following

THEOREM 4.16. *Let A be a ring, let $T = A^2$ or a proper preorder of A. Let $e \in B(A)$ (i.e., an idempotent of A), and let $\rho = \rho_e : \langle A, T \rangle \longrightarrow \langle Ae, Te \rangle$ be the natural p-ring morphism, $\rho(a) = ae$, $a \in A$. For $a \in A$, we write*

$$\begin{aligned} H_A^T(a) &= \{\alpha \in \mathrm{Sper}(A,T) : \pi_\alpha(a) >_\alpha 0\} \\ &= \{\alpha \in \mathrm{Sper}(A,T) : a \in \alpha \setminus \mathrm{supp}(\alpha) = \alpha \setminus -\alpha\} \end{aligned}$$

for the subbasic open determined by a in $\mathrm{Sper}(A,T)$ (endowed with its spectral topology; if $a \in B(A)$, this set is clopen). Then:

a) The properties of being reduced, real or completely real are preserved in passing from A to Ae.

b) The dual (real spectral) map $\mathrm{Sper}(\rho) : \mathrm{Sper}(Ae, Te) \longrightarrow H_A^T(e)$, given by $\alpha \longmapsto \rho^{-1}[\alpha]$, $\alpha \in \mathrm{Sper}(Ae, Te)$, is a homeomorphism. Its inverse is $\beta \longmapsto \beta e$, for $\beta \in H_A^T(e)$.

c) For every $\alpha \in \mathrm{Sper}(Ae, Te)$, *the quotient ring* $(Ae)_\alpha = Ae/\mathrm{supp}(\alpha)$ *and its field of fractions are naturally isomorphic to* $A_{\rho^{-1}[\alpha]}$ *and* $k_{\rho^{-1}[\alpha]}$, *respectively.*

d) If $e_1, \ldots, e_m$ *is an orthogonal decomposition of* A *into idempotents, then:*

(1) $\mathrm{Sper}(A, T)$ *is naturally homeomorphic to the topological sum* $\coprod_{j=1}^m \mathrm{Sper}(Ae_j, Te_j)$;

(2) *For* $a \in A$, $H^T(a) = \coprod_{j=1}^m H^T(ae_j)$. ∎

Lemma 4.18 below makes explicit some facts which occur implicitly in the statement of Theorem 4.16. Preliminarily, we recall the following standard facts on partially ordered rings.

Recall that a **(ring) partial order** (**ring-po**) is a preorder P such that $P \cap -P = \{0\}$.

FACT 4.17. *Let* A *be a ring.*

a) Let T *be a preorder of* A *and consider the following conditions:*

(1) T *is the positive cone of a ring-po of* A.

(2) *The zero ideal is* T*-convex* (cf. 4.14.(a)).

(3) A *is* T*-reduced* (cf. 4.14.4.(a)).

Then, (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1)*; if* A *is reduced, these conditions are equivalent. In particular, if* A *is reduced,* ΣA^2 *is a ring-po iff* A *is real.*

b) Consider the conditions:

(1) A^2 *is the positive cone of a ring-po on* A;

(2) A *is Pythagorean (i.e.,* $A^2 = \Sigma A^2$*) and real.*

Then, (2) $\Rightarrow$ (1)*; if* A *is reduced, these conditions are equivalent.* ∎

LEMMA 4.18. *Let* A *be a ring and let* $T = A^2$ *or a proper preorder of* A. *Set* $Y_T = \mathrm{Sper}(A, T)$ *and let* $B(Y_T)$ *be the Boolean algebra of clopens of* Y_T.

a) If $e \in B(A)$, *then for all* $\alpha \in Y_T$,

$$\pi_\alpha(e) = \begin{cases} 1 & \text{if } e \in \alpha \setminus \mathrm{supp}(\alpha); \\ 0 & \text{if } e \in \mathrm{supp}(\alpha). \end{cases}$$

In particular, $\{H^T(e), H^T(1-e)\}$ *is a clopen partition of* Y_T, *with*

$$H^T(e) = [\![\overline{e} = 1]\!]_T := \{\alpha \in Y_T : \pi_\alpha(e) = 1\} \in B(Y_T).$$

b) The map $H^T : B(A) \longrightarrow B(Y_T)$, *given by* $e \longmapsto H^T(e)$ *is a Boolean algebra morphism.*

c) If T *is a partial order or* $\langle A, T \rangle$ *has* T*-bounded inversion (i.e.,* $1 + T \subseteq A^\times$*), then* H^T *is injective.*

PROOF. a) Since e is a square, we have $e \in \alpha$, for all $\alpha \in Y_T$. Moreover, $\pi_\alpha(e)$ is an idempotent in the integral domain A_α, and so it must be either 0 or 1. Clearly, $\pi_\alpha(e) = 0$ iff $e \in \mathrm{supp}(\alpha)$, establishing the first assertion in (a).

Since $\mathrm{supp}(\alpha)$ ($\alpha \in Y_T$) is a proper prime ideal, we have $e \in \mathrm{supp}(\alpha)$ iff $(1-e) \notin \mathrm{supp}(\alpha)$ which shows Y_T to be the disjoint union of the opens $H^T(e)$ and $H^T(1-e)$, and that hence both are clopen.

b) Clearly, $H^T(0) = \emptyset$, $H^T(1) = Y_T$ and (a) proves that H^T preserves complements ($H^T(1-e) = Y_T \setminus H^T(e)$). To prove that H^T is Boolean algebra morphism it suffices to check preservation of meets. In fact, something more general holds (to be used below):

(†) For $a \in A$ and $e \in B(A)$, $H^T(a) \cap H^T(e) \;=\; H^T(ae)$.

Proof of (†): The equality to be proven amounts to the equivalence

$$\pi_\alpha(a) = 1 \quad \text{and} \quad \pi_\alpha(e) = 1 \quad \Leftrightarrow \quad \pi_\alpha(ae) = 1,$$

for all $\alpha \in Y_T$. Since π_α preserves products, the implication ($\Rightarrow$) is clear, while ($\Leftarrow$) holds because $\pi_\alpha(e)$ is either 0 or 1 (by (a)).

c) Since H^T is a Boolean algebra morphism, it is injective if its kernel is $\{0\}$. Assume $H^T(e) = \emptyset$; then, $\overline{e} = 0$, or equivalently (by 4.14.1 and 4.14.3), $e \in \sqrt[T]{0}$. Thus, there are an integer $k \geq 0$ and $t \in T$ such that

(*) $$e^{2k} + t = e + t = 0.$$

If T is a partial order, (0) is T-convex (cf. Fact 4.17.(a)) and so $e = 0$. If $\langle A, T \rangle$ is a BIR, multiplying (*) by e we obtain $0 = e(1 + t)$, which, together with $1 + t \in A^\times$, yields $e = 0$. ■

Remark 4.19. Even if A is semi-real, it may well happen that for some $e \in B(A)$, Ae *is not* semi-real and so $\mathrm{Sper}(Ae) = \emptyset = H(e)$. Theorem 4.16.(d) also applies to this case. A simple example is the ring $A = \mathbb{R} \times \mathbb{F}_2$, with $e = \langle 0, 1 \rangle$, so that $Ae = (\mathbb{R} \times \mathbb{F}_2)e = \mathbb{F}_2$. However, here we shall only need the preservation result 4.16.(a). ■

CHAPTER 5

First-Order Axioms for Quadratic Faithfulness

We now discuss the first-order axiomatizability of T-quadratic faithfulness. A key to our results lies on the existence of a uniform bound on the length of a witnessing sequence for T-isometry in any T-faithfully quadratic ring, depending only on the dimension of the forms involved. Let $\langle A, T \rangle$ be a p-ring and let φ, ψ be n-forms over a T-subgroup, S, of A. We say that $\langle \varphi, \psi \rangle$ has a witnessing sequence of length $\boldsymbol{k}$ if there are $\varphi = \theta_0, \ldots, \theta_k = \psi$ testifying to $\varphi \approx_T^S \psi$; see Definition 2.17.

Proposition 5.1. *Let $\langle A, T \rangle$ be a p-ring and let S be a T-subgroup of A. If S is T-faithfully quadratic, φ, ψ are forms of dimension n over S and $\varphi \approx_T^S \psi$, then there is a witnessing sequence for this T-isometry of length $\leq \ell(n) = \max\{1, 3(2^{n-1} - 1)\}$.*

Proof. For $n = 1$, the result is trivial and for $n = 2$ it follows from the proof of (2) $\Rightarrow$ (3) in 2.21, showing that $l(2) \leq 3$. We proceed by induction assuming the result true for $n \geq 2$. Let $\varphi = \langle x_0 \rangle \oplus \varphi_1$, $\psi = \langle y_0 \rangle \oplus \psi_1$ satisfy $\varphi \approx_T^S \psi$, where φ_1, ψ_1 have dimension n. If $G_T(S)$ is the SG associated to $\langle S, T \rangle$, as in Theorem 3.6, then by its item (b) we have $\varphi^T \equiv_T^S \psi^T$ in $G_T(S)$. Hence, the definition of isometry in $G_T(S)$ yields u, $v \in S$ and a $(n-1)$-form θ over S such that

$$\text{(I)} \qquad \begin{cases} \langle x_0^T, u^T \rangle \equiv_T^S \langle y_0^T, v^T \rangle\ , \quad \varphi_1^T \equiv_T^S \langle u^T \rangle \oplus \theta^T \quad \text{and} \\ \qquad\qquad \psi_1^T \equiv_T^S \langle v^T \rangle \oplus \theta^T. \end{cases}$$

Now (I) and 3.6.(b) yield

$$\text{(II)} \qquad \langle x_0, u \rangle \approx_T^S \langle y_0, v \rangle \quad \varphi_1 \approx_T^S \langle u \rangle \oplus \theta \quad \text{and} \quad \psi_1 \approx_T^S \langle v \rangle \oplus \theta.$$

It follows from the *proof* of 2.19.(d.1) that if β, η are forms of the same dimension over S, whose T-isometry has a witnessing sequence of length k, and λ is a form over S, then

$$(*) \qquad \lambda \oplus \beta \approx_T^S \lambda \oplus \eta \text{ has a witnessing sequence of length } k.$$

Since $l(2) \leq 3$, the first T-isometry in (II), 2.19.(d.1) and (*) above entail that $\langle x_0, u \rangle \oplus \theta \approx_T^S \langle y_0, v \rangle \oplus \theta$ has a witnessing sequence of length ≤ 3. Similarly, the last two T-isometries in (II), the induction hypothesis and (*) yield witnessing sequences of length at most $\ell(n)$ for

$$\varphi = \langle x_0 \rangle \oplus \varphi_1 \quad \approx_T^S \quad \langle x_0, u \rangle \oplus \theta \quad \text{and} \quad \langle y_0, v \rangle \oplus \theta \quad \approx_T^S \langle y_0 \rangle \oplus \psi_1 = \psi.$$

By concatenation of these sequences we obtain a witnessing sequence for $\langle \varphi, \psi \rangle$ of length at most

$$2\ell(n) + 3 \ = \ 2(3(2^{n-1} - 1)) + 3 \ = \ 3(2^n - 1) \ = \ \ell(n+1),$$

as desired. ■

The notions of geometrical and Horn-geometrical formula and theory appear in items (e) and (g) of Definition 1.4.

THEOREM 5.2. *a) The theory of faithfully quadratic rings is Horn-geometrical in the language of unitary rings, having as operations and constants the set* $\{+, \cdot, 0, 1, -1\}$.

b) The theory of T-faithfully quadratic rings is geometrical in the language of unitary rings, together with a unary predicate symbol, T, which stands for a preorder.

PROOF. The reader should keep in mind the observations in 1.5.(d). We shall present a proof of (b), indicating the modifications to obtain (a).

To formulate the theory of T-faithfully quadratic rings, we must add to the axioms for p-rings (which are Horn-geometric, cf. Definition 6.1 in [**DM8**] or 2.1.(c)), first-order sentences corresponding to the axioms [T-FQ i] in 3.1, $i = 1, 2, 3$. It will become clear from the proof that the theory of faithfully quadratic rings is Horn-geometrical, while, because of axiom [T-FQ 3], T a preorder, our argument gives explicit geometrical axioms for the theory of T-quadratic faithfulness.[1]

We shall indicate forms by the initial letters of the Greek alphabet, reserving φ, ψ, σ for formulas in the first-order language L_T whose non-logical symbols are $\{+, \cdot\ , T, 0, 1, -1\}$ (or $\{+, \cdot\ , 0, 1, -1\}$ for the case of square classes).

For each $n \geq 1$:

(I) Let $(\varphi_n)^v_T(a, b_1, \ldots, b_n)$ be the formula in $n+1$ free variables given by

$$\exists\, u, w_1, \ldots, w_n, t_1, \ldots, t_n \ (au = 1 \ \wedge \ \textstyle\bigwedge_{j=1}^n w_j b_j = 1 \ \wedge \ \bigwedge_{j=1}^n t_j \in T \ \wedge \ a = \sum_{j=1}^n t_j b_j).$$

Clearly, $(\varphi_n)^v_T(a, \overline{b})$ is a pp-formula in L_T and its meaning is that $a, b_1, \ldots, b_n \in A^\times$ and $a \in D^v_T(b_1, \ldots, b_n)$. In the case of A^2, we may omit mention to T and replace the last two conjuncts by $a = \sum_{j=1}^n t_j^2 b_j$.

(II) Let $(\varphi_n)^t_T(a, b_1, \ldots, b_n)$ be the formula in $n+1$ free variables given by

$$\exists\, u, w_1, \ldots, w_n, t_1, \ldots, t_n, z_1, \ldots, z_n \ (au = 1 \ \wedge \ \textstyle\bigwedge_{j=1}^n w_j b_j = 1 \ \wedge \ \bigwedge_{j=1}^n z_j t_j = 1 \ \wedge \ \bigwedge_{j=1}^n t_j \in T \ \wedge \ a = \sum_{j=1}^n t_j b_j).$$

Clearly, $(\varphi_n)^t_T(a, \overline{b})$ is a pp-formula in L_T, expressing $a, b_1, \ldots, b_n \in A^\times$ and $a \in D^t_T(b_1, \ldots, b_n)$. In the case A^2 we may omit mention to T and replace the last two conjuncts by $a = \sum_{j=1}^n t_j^2 b_j$.

We can now give Horn-geometric axioms for [T-FQ 1] and [T-FQ 2], as follows:

[1] This theory will be shown to be Horn in Corollary 5.6.

• For [T-FQ 1], let $\boldsymbol{\tau}$ be the sentence

$$\forall\, a,\, b_1,\, b_2\; ((\varphi_2)^v_T(a,\, b_1,\, b_2) \;\;\rightarrow\;\; (\varphi_2)^t_T(a,\, b_1,\, b_2)),$$

which expresses "$a \in D^v_T(b_1, b_2)$ *implies* $a \in D^t_T(b_1, b_2)$". Similarly, for [FQ 1]. By 1.5.(d), $\boldsymbol{\tau}$ is a Horn-geometric sentence in L_T. It is also clear that T-transversality of n-representation is rendered by a Horn-geometric formula in L_T, $n \geq 2$.

• To obtain [T-FQ 2] we add, for each $n \geq 2$, the sentence $\boldsymbol{\sigma_n}$ (with the standing convention that $\check{w}$ indicates omission of that variable):

$$\bigwedge_{k=1}^{n} \forall\, a,\, \overline{b}\; ((\varphi_n)^v_T(a, \overline{b}) \;\;\rightarrow\;\; \exists\, w\, w'\; (ww' = 1 \;\;\wedge\;\; (\varphi_2)^v_T(a,\, b_k,\, w)$$
$$\wedge \;\; (\varphi_{n-1})^v_T(w,\, b_1,\, \ldots,\, \check{b}_k,\, \ldots,\, b_n))),$$

that describes "$a \in D^v_T(b_1, \ldots, b_n)$ *implies* $a \in \mathfrak{D}_T(b_1, \ldots, b_n)$". Since in any p-ring $\mathfrak{D}_T(\varphi) \subseteq D^v_T(\varphi)$ (2.26.(c)), the collection $\{\sigma_n \,:\, n \geq 2\}$ is a set of Horn-geometric sentences rendering [T-FQ 2] in L_T (by items (a), (c) and (d) in 1.5). The modifications to obtain the Horn-geometric expression of [FQ 2] are clear.

For [T-FQ 3], recall (5.1) that $\ell(n) = \max\,\{1,\, 3(2^{n-1}\, - 1)\}$, for each $n \geq 2$. All we need is to express $\alpha \, \approx_T \, \beta$ by geometric sentences, where α, β are n-forms over $A^\times$. We first make the following observations:

(III) To ease presentation, if $\overline{a}$, $\overline{b}$ are n-tuples of variables, write $\overline{a} \cdot \overline{b} = (a_1 b_1, \ldots, a_n b_n)$, and write $\overline{1}$ for the constant n-sequence whose entries are all equal to 1. Thus, $\overline{a} \cdot \overline{b} = \overline{c}$ is equivalent to $\bigwedge_{j=1}^{n} a_j b_j = c_j$.

(IV) Let α, β be n-forms over $A^\times$. If $\alpha = \alpha_0, \ldots, \alpha_k = \beta$ is a witnessing sequence for $\alpha \approx_T \beta$, then to each $1 \leq j \leq k$ we associate a pair $\langle\, M_j, \overline{t}_j \,\rangle \in \mathrm{GL}_n(A) \times T^{\times n}$ such that

$$\alpha_{j-1} \cdot \overline{t}_j = \alpha_j \quad \textbf{or} \quad M_j\; \mathscr{M}(\alpha_j)\; M_j^t = \mathscr{M}(\alpha_{j-1}),$$

as follows:

• If at step j we have $N \in \mathrm{GL}_n(A)$ such that $N\; \mathscr{M}(\alpha_j)\; N^t = \mathscr{M}(\alpha_{j-1})$, then $M_j = N$ and $\overline{t}_j = \overline{1}$;

• If at step j we have $\alpha_j = \alpha_{j-1} \cdot \overline{s}$, with $\overline{s} \in T^{\times n}$, take $M_j = Id_n$ (the identity matrix in $\mathrm{GL}_n(A)$) and $\overline{t}_j = \overline{s}$.

Clearly, the pairs $\langle\, M_j, \overline{t}_j \,\rangle$, $1 \leq j \leq k$, satisfy the required condition.

(V) Recall the following standard facts:

• Matrices of size $n \times n$ ($n \geq 1$) with variable coefficients are coded in the language of rings as sequences of n^2 distinct variables. For fixed $n \geq 1$,

$$\exists\, M\; (***) \quad \text{stands for} \quad \exists\, v_{11} \cdots v_{1n}\; v_{21} \cdots v_{nn}\; (***),$$

where M is the $n \times n$ matrix $\begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix}$.

• The product $MP = R$ of $n \times n$ matrices with variable coefficients is a conjunction of n^2 equalities in the language of rings.

• The assertion that a $n \times n$ matrix is diagonal is a conjunction of n^2 equalities in the language of rings.

• The assertion that a $n \times n$ matrix, M, is in $\mathrm{GL}_n(A)$ is expressed by a pp formula in the language of rings, written "det M *invertible*".

Consider, the formula $\psi^{nk}(\overline{x},\ \overline{y})$, $n,\ k \geq 1$ and $\overline{x} = \langle x_1, \ldots, x_n \rangle$, $\overline{y} = \langle y_1, \ldots, y_n \rangle$, given by

$$\exists\ \alpha_0,\ \alpha_1, \ldots, \alpha_k,\ \overline{a}_0,\ \overline{a}_1, \ldots, \overline{a}_k,\ M_1, \ldots, M_k,\ \overline{t}_1, \ldots, \overline{t}_k,$$

$$\exists\ \overline{w}_1, \ldots, \overline{w}_k \Big(\ \overline{x} = \alpha_0 \ \wedge\ \overline{y} = \alpha_k \ \wedge\ \textstyle\bigwedge_{j=0}^{k} \alpha_j \cdot \overline{a}_j = \overline{1} \ \wedge$$

$$\wedge\ \textstyle\bigwedge_{j=1}^{n} \overline{t}_j \cdot \overline{w}_j = \overline{1} \ \wedge\ \textstyle\bigwedge_{j=1}^{n} \overline{t_j} \in T^n \ \wedge\ \textstyle\bigwedge_{j=1}^{n} \text{"det } M_j \text{ invertible"} \ \wedge$$

$$\wedge\ \underline{\textstyle\bigwedge_{j=1}^{n} \left(M_j\, \mathscr{M}(\alpha_j)\, M_j^t \ =\ \mathscr{M}(\alpha_{j-1}) \ \vee\ \alpha_j \ =\ \alpha_{j-1} \cdot \overline{t}_j\right)\Big)},$$

where each of $\alpha_0,\ \alpha_1, \ldots, \alpha_k,\ \overline{a}_0,\ \overline{a}_1, \ldots, \overline{a}_k,\ \overline{t}_1, \ldots, \overline{t}_k,\ \overline{w}_1, \ldots, \overline{w}_k$ are n-tuples of variables and $M_1, \ldots, M_k$ are n^2-tuples of variables, all distinct; $\psi^{nk}(\overline{x},\ \overline{y})$ asserts

"$\overline{x} = \alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{k-1},\ \alpha_k = \overline{y}$ *is a witnessing sequence for* $\langle\, \overline{x}\, \rangle \approx_T \langle\, \overline{y}\, \rangle$".

Because of the disjunction in the underlined formula above, $\psi^{nk}(\overline{x},\ \overline{y})$ is a *geometrical* formula in L_T (but not necessarily Horn!). If $T = A^2$, the underlined formula can replaced by a simpler one, expressing $\langle\, \overline{x}\, \rangle \approx \langle\, \overline{y}\, \rangle$, namely:

"*All entries of* $\overline{x}$ *and* $\overline{y}$ *are units and there is an invertible matrix* M*,*

such that $M\ \mathscr{M}(\overline{x})\ M^t = \mathscr{M}(\overline{y})$)",

clearly a positive primitive formula in the language of rings. Hence, in case $T = A^2$, $\psi^{nk}(\overline{x}, \overline{y})$ is a pp formula in the language of rings.

Now, for each pair $\langle\, n, k\, \rangle$, $n,\ k \geq 1$, we adjoin the sentence $\boldsymbol{\Psi^{nk}}$

$$\forall a,\ \overline{x},\ \overline{y},\ u\ ((ua = 1 \ \wedge\ \psi^{(n+1)k}(\langle\, a, \overline{x}\, \rangle, \langle\, a, \overline{y}\, \rangle) \ \to\ \psi^{n\ell(n)}(\overline{x},\ \overline{y})),$$

with $\ell(n)$ as in Proposition 5.1. By items (c) and (d) in 1.5, $\boldsymbol{\Psi^{nk}}$ is geometrical in L_T, and in case $T = A^2$, Horn-geometrical in the language of rings. Let $\mathcal{T}$ be the theory in L_T consisting of the union of the following two sets of L_T-sentences:

• The axioms for unitary p-rings wherein 2 is a unit, and T is a preorder of A; if we wish to restrict our constructions to *proper* p-rings, we add $-1 \notin T$, the negation of an atomic formula;

• The set of L_T-sentences $\{\tau\} \cup \{\sigma_n : n \geq 2\} \cup \{\Psi^{nk} : n,\ k \geq 1\}$.

Then, $\mathcal{T}$ is a geometrical theory in L_T, all of whose models are T-faithfully quadratic rings, in view of the contents of its axioms.

Conversely, suppose $\langle\, A, T\, \rangle$ is a T-faithfully quadratic p-ring. Then, $\langle\, A, T\, \rangle \models \tau$ (by [T-FQ 1]) and $\langle\, A, T\, \rangle \models \sigma_n$, for all $n \geq 2$ (by [T-FQ 2]). With respect to $\boldsymbol{\Psi^{nk}}$, assume that for $a \in A^{\times}$ and $\overline{x},\ \overline{y} \in {A^{\times}}^n$, there is a witnessing sequence of length k for $\langle\, a\, \rangle \oplus \langle\, \overline{x}\, \rangle \approx_T \langle\, a\, \rangle \oplus \langle\, \overline{y}\, \rangle$; since $\langle\, A, T\, \rangle$ is T-faithfully quadratic, Proposition 5.1 yields a witnessing sequence for $\langle\, \overline{x}\, \rangle \approx_T \langle\, \overline{y}\, \rangle$ of length $m \leq \ell(n)$; if $m < \ell(n)$, we obtain a witnessing sequence of exact length $\ell(n)$ simply by repeating $(\ell(n) - m)$ times the form $\langle\, \overline{y}\, \rangle$. Hence, $\langle\, \overline{x}\, \rangle \approx_T \langle\, \overline{y}\, \rangle$ has a witnessing of length precisely $\ell(n)$ and $\langle\, A, T\, \rangle \models \boldsymbol{\Psi^{nk}}$. Similar arguments show that, if $T = A^2$, the corresponding Horn-geometrical sentences axiomatize the theory of faithfully quadratic rings, concluding the proof. ■

REMARK 5.3. With minor adaptations in the proof of Theorem 5.2, one can show that the theories of faithfully quadratic q-subgroups of a ring and of T-faithfully quadratic T-subgroups of a ring are Horn-geometrical and geometrical, respectively, in the language of rings, now augmented by a new unary predicate S to be interpreted as a q-subgroup or T-subgroup of the ring, respectively. To fix ideas, we describe the modifications needed for T-quadratic faithfulness of T-subgroups. We consider the language $L_{S,T}$ with non-logical symbols $\{+, \cdot, T, S, 0, 1, -1\}$, and proceed as follows:

• In the formulas τ, σ_n, $n \geq 2$, and ψ^{nk}, n, $k \geq 1$, we must specify that the variables are in S;

• We must also include the Horn-geometric sentences:

(1) $S(-1) \wedge \forall\ x\ (S(x) \to \exists\ u\ (ux = 1))$;

(2) $\forall\ x\ \forall y\ (S(x) \wedge S(y) \to S(xy))$;

(3) $\forall x\ \forall u\ ((T(x) \wedge ux = 1) \to S(x))$,

expressing that $-1 \cup T^{\times} \subseteq S \subseteq A^{\times}$ and that S is closed under products. In the case of A^2, sentence (3) is to be replaced by

(3′) $\forall x\ \forall u\ (ux = 1 \to S(x^2))$. ■

Although the results that follow hold true for T-faithfully quadratic T-subgroups, they will be stated, to ease presentation, just for the case of p-rings.

COROLLARY 5.4. *Let $\mathscr{A} = \langle\, \langle\, A_\lambda, T_\lambda \,\rangle; \{f_{\lambda\mu} : \lambda \leq \mu \text{ in } \Lambda\}\,\rangle$ be an inductive system of T_λ-faithfully quadratic p-rings over the right-directed poset $\langle\, \Lambda, \leq \,\rangle$. If $\langle\, A, T \,\rangle = \varinjlim \mathscr{A}$ is the inductive limit of $\mathscr{A}$, then A is T-faithfully quadratic and $G_T(A) = \varinjlim_{\lambda\in\Lambda} G_{T_\lambda}(A_\lambda)$. A similar statement holds for faithfully quadratic rings.*

PROOF. It is well-known that the class of models of a geometric theory is closed under right-directed inductive limits (cf. [**Mir**], section 4, chapter 17), yielding the first assertion.[2]

The characterization of the π-SG associated to the inductive limit follows from Proposition 2.9. ■

PROPOSITION 5.5. *Let $\{\langle\, A_i, T_i \,\rangle : i \in I\}$ be a non-empty family of p-rings and let D be a filter on I. Let $\langle\, A_D, T_D \,\rangle = \langle\, \prod_D A_i, \prod_D T_i \,\rangle$ be the reduced product of the $\langle\, A_i, T_i \,\rangle$ modulo D. If for all $i \in I$, A_i is T_i-faithfully quadratic, then A_D is T_D-faithfully quadratic and $G_{T_D}(A_D) = \prod_D G_{T_i}(A_i)$. An analogous statement holds for faithfully quadratic rings.*

PROOF. In the case of quadratic faithfulness, the result follows from Proposition 6.6.2 in [**CK**], guaranteeing that Horn theories – and, *a fortiori*, Horn-geometrical ones –, are preserved by reduced products.

For the case of p-rings, we argue as follows:

• By Theorem 4.6, the theory of T-faithfully quadratic rings is preserved by arbitrary products;

[2] In fact, theories preserved under right-directed inductive limits possess a geometrical axiomatization, cf. [**Ki**].

• By Fact 2.10, $\langle A_D, T_D \rangle$ is a right-directed inductive limit of the products $A(U) = \prod_{i \in U} A_i$, for $U \in D$,

and hence Corollary 5.4 entails the T_D-quadratic faithfulness of A_D. The last assertion is a consequence of Proposition 2.9. ■

An interesting consequence of Proposition 5.5 is:

COROLLARY 5.6. *The theory of T-faithfully quadratic rings, where T is a preorder, has a Horn axiomatization.*

PROOF. A deep result due to Kiesler, Galvin and Shelah (Theorem 6.2.5′, p. 366, [**CK**]) shows that a first-order theory is preserved by *arbitrary* reduced products iff it is Horn-axiomatizable. The desired conclusion follows immediately from this and Proposition 5.5. ■

The geometrical axiomatization of the theory of T-faithfully quadratic rings (T a preorder) presented in Theorem 5.2 fails, as noted, to be Horn; it would be interesting to determine an *explicit* Horn axiomatization of this theory.

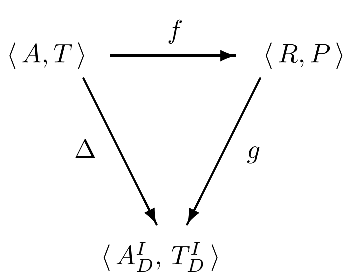
With notation as in 2.8 and 1.4.(d), we have

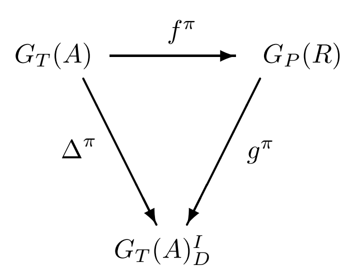
LEMMA 5.7. *a) If $f : \langle A, T \rangle \longrightarrow \langle R, P \rangle$ is a pure p-ring morphism, then $f^\pi : G_T(A) \longrightarrow G_P(R)$ is a pure π-SG morphism.*

b) If $\langle A, T \rangle$ is elementary equivalent to $\langle R, P \rangle$, the same is true of their associated π-SGs. Similar statements hold if $T = A^2$ and $P = R^2$.

c) Suppose $\langle A_2, T_2 \rangle$ is T_2-faithfully quadratic and let $f : \langle A_1, T_1 \rangle \longrightarrow \langle A_2, T_2 \rangle$ be a p-ring morphism that does not necessarily preserve 1. If f is pure in the language $\langle +, \cdot, T, 0 \rangle$, where T is a unary predicate standing for the preorders on A_i, $i = 1, 2$, then A_1 is T_1-faithfully quadratic. Moreover, if $e = f(1)$, then e is an idempotent in R and $f^\pi : G_{T_1}(A_1) \longrightarrow G_{T_2 e}(A_2 e)$ is a pure SG-morphism. A similar statement holds for faithfully quadratic rings (no need of the predicate T).

PROOF. a) By Proposition 1.4.(b) in [**DM4**] (see also Proposition on p. 92 of [**DM2**]), the purity of f is equivalent to the existence of a set I, an ultrafilter D on I and a p-ring morphism, g, such that the diagram below left is commutative, where Δ is the canonical diagonal embedding of $\langle A, T \rangle$ into its ultrapower $\langle A^I_D, T^I_D \rangle$





Since the functor from p-rings to π-SGs preserves arbitrary reduced products (Proposition 2.9), it follows that its application to the diagram above left yields the commutative diagram of π-SG morphisms above right, where Δ^π is clearly the canonical diagonal embedding of $G_T(A)$ into $G_T(A)^I_D$; another application of Proposition 1.4.(b) in [**DM4**] yields the desired conclusion.

The proof of item (b) is similar, using the Keisler-Shelah ultrapower theorem (Theorem 6.1.5, [**CK**]) in place of Proposition 1.4 in [**DM4**].

c) Clearly $e = f(1)$ is an idempotent in A_2; hence, $f : \langle A_1, T_1 \rangle \longrightarrow \langle A_2 e, T_2 e \rangle$ is pure, but now in the language of *unitary* rings with the additional unary predicate T and constant -1 (-1 in $A_2 e$ is $-e$). The T_1-quadratic faithfulness of A_1 follows from Lemma 1.6 and Corollary 4.7, while the statement concerning the associated *special groups* follows from item (a). ■

CHAPTER 6

Rings with Many Units

In this chapter we show that:

• The class of rings with many units can be axiomatized by Horn-geometric sentences, reaping some interesting consequences of this fact.

• A large class of rings with many units is **completely faithfully quadratic**, i.e., faithfully quadratic and T-faithfully quadratic for all preorders T. Moreover, if T is a proper preorder of a ring A in this class, the reduced special group $G_T(A)$ is *isomorphic* to the group of units of the real semigroup $\mathscr{G}_{A,T}$ (cf. [**DP1**], [**DP2**] and 4.10 below).

1. Rings with Many Units are Horn-Geometrical

We start by recalling

Definition 6.1. *Let R be a ring.*

a) A polynomial $f \in R[X_1, \ldots, X_n]$ with coefficients in R **has local unit values** *if for every maximal ideal $\mathfrak{m}$ of R, there are $u_1, \ldots, u_n$ in R such that $f(u_1, \ldots, u_n)$ is not in $\mathfrak{m}$.*

b) R is a **ring with many units** *if for all $n \geq 1$, and all $f \in R[X_1, \ldots, X_n]$, if f has local unit values, then there are $r_1, \ldots, r_n \in R$ such that $f(r_1, \ldots, r_n) \in R^{\times}$.* ■

Remark 6.2. a) For more information on rings with many units the reader is referred to [**Mc**] (therein called *local-global rings*), [**McW**], [**MW**], [**Mar**] (p. 153), [**Wa**], section 4 of [**DM7**] and [**DM9**]. Fields, semi-local rings, von Neumann regular rings and arbitrary products of rings with many units have many units.

b) The last two classes of examples mentioned in (a) are special cases of Theorem 3.5 in [**DM9**] : the ring of global sections of a sheaf of rings over a partitionable space, whose stalks are rings with many units, is also a ring with many units. In this respect, see also Corollary 3.6 in [**DM9**]. Hence, the ring of global sections of a sheaf of rings over a Boolean space, whose stalks are local rings, is a ring with many units. In particular, the ring of continuous real-valued functions on a Boolean space has many units.

c) In section 3 below, we give new examples of rings with many units, namely rings of formal power series over a ring with many units. ■

Although we have a blanket assumption that in all rings herein considered 2 is a unit, this hypothesis is not needed for the proof of our next result.

THEOREM 6.3. *Rings with many units are axiomatizable by Horn-geometric sentences in the language of unitary rings with equality.*

PROOF. With notation as in 6.1, we first note the following straightforward

Fact. *If A is a ring and $f \in A[X_1, \ldots, X_n]$, then the following are equivalent:*

(1) *f has local unit values;*

(2) *There are an integer $k \geq 1$, $\overline{a}_1, \ldots, \overline{a}_k \in A^n$ and $\lambda_1, \ldots, \lambda_k \in A$ such that $\sum_{j=1}^{k} \lambda_j f(\overline{a}_j) \ = \ 1$, that is, the ideal generated by $\{f(\overline{a}) : \overline{a} \in A^n\}$ is A.* □

For integers $n,\, d \geq 1$, let

$$\mathcal{V}(n,\, d) \ = \ \{\nu = \langle\, \nu_1, \ldots, \nu_n\,\rangle \in \{0, 1, \ldots, d\}^n : \textstyle\sum_{i=1}^{n} \nu_i \leq d\}.$$

The cardinal of $\mathcal{V}(n,\, d)$ is $\binom{n+d}{d}$, exactly the number of monomials in n variables of *total* degree $\leq d$. Let L be the first-order language of unitary rings. To simplify notation, if $\overline{x}_\nu = \langle\, x_\nu\,\rangle_{\nu \in \mathcal{V}(n,d)}$ and $\overline{X} = \langle\, X_1, \ldots, X_n\,\rangle$ are distinct variables, let $\tau(\overline{x}_\nu;\, \overline{X})$ be the term in $\binom{n+d}{d} + n$ free variables of L given by

$$\tau(\overline{x}_\nu;\, \overline{X}) \ = \ \textstyle\sum_{\nu \in \mathcal{V}(n,d)} x_\nu X_1^{\nu_1} \cdots X_n^{\nu_n}.$$

For integers $n,\, d,\, k \geq 1$, let $\sigma(n, d, k)$ be the following sentence in L:

$$\sigma(n,d,k) := \forall\, \overline{x}_\nu \ \Big(\Big(\exists \overline{Z}_1 \ldots \overline{Z}_k \ \exists\, \lambda_1 \ldots \lambda_k \ \Big(\textstyle\sum_{j=1}^{k} \ \lambda_j \tau(\overline{x}_\nu; \overline{Z}_j) \ = \ 1 \Big) \Big) \rightarrow \ \exists u \exists \overline{Y} \left(u \cdot \tau(\overline{x}_\nu; \overline{Y}) = \ 1 \right) \Big).$$

Clearly, each $\sigma(n,d,k)$ is a Horn-geometrical sentence of L (cf. 1.4.(g)). Let $\mathcal{T}$ be theory in L consisting of the axioms for commutative unitary rings, together with $\{\sigma(n,d,k) \,:\, n,\, d,\, k \geq 1\}$; then, $\mathcal{T}$ is a Horn-geometrical theory in L, which we claim to be the theory of rings with many units. Indeed :

• If A is a model of $\mathcal{T}$, then A is a commutative unitary ring. Moreover, a polynomial $f \in A[X_1, \ldots, X_n]$, of total degree d, gives rise to a term $\tau(\overline{c}_\nu;\, \overline{X})$, where $\overline{c}_\nu$ are the $\binom{n+d}{d}$ coefficients of the monomials in f. Assume that f has local unit values. By the Fact above, for some $k \geq 1$ and $\overline{a}_1, \ldots, \overline{a}_k \in A^n$, the antecedent of the implication in $\sigma(n,d,k)$ holds true in A, with $\overline{Z}_k$ substituted by $\overline{a}_k$, and hence its consequent guarantees the existence of $\overline{b} \in A^n$ such that $\tau(\overline{c}_\nu;\, \overline{b}) = f(\overline{b}) \in A^{\times}$, showing that A has many units.

• Let A be a ring with many units, let $n,\, d,\, k \geq 1$ be integers and fix $\{\overline{c}_\nu : \nu \in \mathcal{V}(n,d)\} \subseteq A$. The term $\tau(\overline{c}_\nu;\, \overline{X})$ is then a polynomial, $f \in A[X_1, \ldots, X_n]$. To show that $A \models \sigma(n,d,k)$, we assume that the antecedent of the implication in $\sigma(n,d,k)$ holds in A. By the Fact above, this means that f has local unit values in A and so there is $\overline{b} \in A^n$ such that $f(\overline{b}) \in A^{\times}$, precisely the content of the consequent of the implication in $\sigma(n,d,k)$, completing the proof. ■

From Theorem 6.3, we obtain

COROLLARY 6.4. *The class of rings with many units is closed under arbitrary reduced products—in particular, ultraproducts—and arbitrary inductive limits over right-directed posets.*

PROOF. The result follows from Theorem 6.3, recalling, as noted in the proof of Corollary 5.4, that Horn-geometrical theories are preserved under reduced products and right-directed colimits. ∎

2. Rings with Many Units and Quadratic Faithfulness

In this section we prove that a large class of rings with many units are completely faithfully quadratic. The results needed to verify our axioms appear in Walter's thesis [**Wa**], some of which are written in the classical language of quadratic spaces over finite dimensional free A-modules, but are straightforwardly translatable to our setting. We also mention Theorem 8.1.7 (p. 154) of [**Mar**], which in the language of real-semigroups ([**DP1**], [**DP2**]) implies that if $\langle A, T \rangle$ is a p-ring with many units, the group of units, $\mathscr{G}^*_{A,T}$, of the real-semigroup $\mathscr{G}_{A,T}$ associated to $\langle A, T \rangle$, is a special group. In this regard, we note that, the aforementioned result does not cover the case of square classes, and to establish the complete quadratic faithfulness of $\mathscr{G}^*_{A,T}$ *it is not enough* to show it to be a special group (cf. Example 10.14). However, by Proposition 6.9 below, if $\langle A, T \rangle$ is T-faithfully quadratic and A has many units, the groups $G_T(A)$ and $\mathscr{G}^*_{A,T}$ are isomorphic, yielding, for this class of rings, a new proof that the latter is a reduced special group.

The following result first appeared as Theorem 3.16 (pp. 17–18) in [**DM6**], with an entirely different and much lengthier proof.

THEOREM 6.5. *If A is a ring with many units such that every residue field of A modulo a maximal ideal has at least 7 elements, then A is completely faithfully quadratic.*

PROOF. We will first show that A is faithfully quadratic and then use Theorem 3.9.(b) to prove that A is T-faithfully quadratic, for all preorders T on A. To show that A is faithfully quadratic, we employ results appearing in [**Wa**], in which the hypothesis that the residue fields of A modulo maximal ideals have all at least seven elements is needed in certain counting arguments. Notation is as in Definition 2.24.

[FQ 1] : This follows from Proposition 6.1 in [**Wa**] (p. 25), guaranteeing that if all residue fields of A have at least 7 elements, then for all forms φ with coefficients in $A^\times$, $D^v(\varphi) = D^t(\varphi)$.

[FQ 2] : This follows from Theorem 4.3 and its Corollary 4.4 in [**Wa**] (pp. 21–22), showing that if φ is a form with coefficients in $A^\times$, then $\mathfrak{D}(\varphi) = D^v(\varphi)$.

[FQ 3]: This is a consequence of Theorem 4.1 in [**Wa**] (p. 21) establishing that $\varphi \oplus \psi \approx \varphi \oplus \theta$ entails $\psi \approx \theta$ (Witt cancellation).

Now, let T be a preorder of A. By the equivalence in Theorem 3.9.(b), to prove that A is T-faithfully quadratic it suffices to establish:

$$\text{(I)} \quad \begin{cases} \text{For all } x, a_1, \ldots, a_n \in A^\times, \ \text{ if } x \in D^v_T(a_1, \ldots, a_n), \text{ then there are } x_2, \ldots, \\ x_n \in A^\times \text{ and } t_1, \ldots, t_n \in T^\times \text{ such that } \langle x, x_2, \ldots, x_n \rangle \approx \langle t_1a_1, \ldots, t_na_n \rangle. \end{cases}$$

In Proposition 5.5 of [**Wa**] (p. 43; this result is akin to Theorem 3.5 in [**MW**]) it is shown that if $x \in D^v_T(a_1, \ldots, a_n)$, then there are $t_1, \ldots, t_n \in T^\times$ such that $x \in D^v(t_1a_1, \ldots, t_na_n)$. By Corollary 1.5 in [**Wa**] (p. 16), there are $x_2, \ldots, x_n \in A^\times$

such that $\langle x, x_2, \dots, x_n \rangle \approx \langle t_1a_1, \dots, t_na_n \rangle$, as needed to establish (I), ending the proof. ■

From Theorem 2.16 (or Corollary 3.13) we get

COROLLARY 6.6. *If A is a ring with many units satisfying the assumptions of Theorem 6.5, then Milnor's mod 2 K-theory of A is isomorphic to the K-theory of the special group $G(A)$.* ■

Given a p-ring $\langle A, T \rangle$, we now turn to the question of the relation between $G_T(A)$ and the group of units of the real semigroup associated to $\langle A, T \rangle$. For the convenience of the reader, we summarize below the basic facts and notation to be used in the next result and in section 2 of Chapter 7.

6.7. **Preliminaries and Notation** [Real Semigroups]. With notation as set in 4.10,

a) Let $\mathscr{G}_{A,T} = \{\overline{a} : a \in A\}$. If $T = \Sigma A^2$, we write $\mathscr{G}_A$ for $\mathscr{G}_{A,T}$. Clearly, $\mathscr{G}_{A,T}$ is a commutative semigroup (monoid) containing the constant functions with values 1, 0, -1, and such that $\overline{a}^3 = \overline{a}$, for all $a \in A$. Thus, $\mathscr{G}_{A,T}$ is a **ternary semigroup** in the sense of section 1 of [**DP1**], and section 1 of Chapter 1 of [**DP2**]. Further, $\mathscr{G}_{A,T}$ separates points in Sper(A, T): for $\alpha, \beta \in$ Sper(A, T), $\alpha \neq \beta$, there is $a \in A$ so that $\overline{a}(\alpha) \neq \overline{a}(\beta)$; cf. [**Mar**], Theorem 6.1.2, p. 100.

Set $\mathscr{G}^*_{A,T} = \{\overline{a} \in \mathscr{G}_{A,T} : \overline{a}^2 = 1\}$, called the **group of units** of $\mathscr{G}_{A,T}$, consisting of the $\overline{a}$ whose value at every $\alpha \in$ Sper(A, T) is distinct from 0. Clearly, $\mathscr{G}^*_{A,T}$ is a multiplicative group of exponent two.

b) The semigroup $\mathscr{G}_{A,T}$ carries two 'representation' relations, defined in [**Mar**], pp. 95, 96. To distinguish them from the other − closely related − relations defined in Chapter 2 (cf. 2.24), we shall call them the **value set** and the **transversal value set** of a form with entries in $\mathscr{G}_{A,T}$ (see item (5) in Proposition 5.5.1 (p. 95) and the paragraph preceding the Note on p. 96 of [**Mar**]).

DEFINITION 6.8. *Given a p-ring $\langle A, T \rangle$, $a_1, \dots, a_n$, $b \in A$ and a form $\overline{\varphi} = \langle \overline{a}_1, \dots, \overline{a}_n \rangle$ over $\mathscr{G}_{A,T}$, we set*

(i) $\overline{b} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{\varphi}) \;\Leftrightarrow\; \exists\, t, t_1, \dots, t_n \in T$ *such that* $tb = \sum_{i=1}^n t_ia_i$ *and* $\overline{t}\,\overline{b} = \overline{tb} = \overline{b}$;

(ii) $\overline{b} \in \mathscr{D}^t_{\mathscr{G}_{A,T}}(\overline{\varphi}) \;\Leftrightarrow\; \exists\, b', a'_1, \dots, a'_n$ *such that* $\overline{b} = \overline{b'}$, $\overline{a_i} = \overline{a'_i}$, $1 \leq i \leq n$, *and* $b' = \sum_{i=1}^n a'_i$. ■

Equipped with any of these relations, $\mathscr{G}_{A,T}$ is a **real semigroup** in the sense of section 1 of [**DP1**], or section 2 of Chapter 1 in [**DP2**].

c) Let X be a set and $\langle L, \leq \rangle$ be a partially ordered set. For $f : X \longrightarrow L$ and $a \in L$, define

$$[\![f = a]\!] = \{x \in X : f(x) = a\} \quad \text{and} \quad [\![f < a]\!] = \{x \in X : f(x) < a\}.$$

Similarly, one defines $[\![f \leq a]\!]$ and $[\![f \geq a]\!]$. For instance, if $\langle A, T \rangle$ is a p-ring, Y_T = Sper(A, T) and $L = \{-1, 0, 1\}$, then for $a \in A$ and with notation as in 4.10, and recalling that (cf. 4.16)

$$H^T_A(x) = \{\alpha \in Y_T : x \in \alpha \setminus -\alpha\} = \{\alpha \in Y_T : \pi_\alpha(x) >_\alpha 0\},$$

we have

$$(*)\qquad \begin{cases} [\![\overline{a} < 0]\!] &= [\![\overline{a} = -1]\!] = H_A^T(-a); \\ [\![\overline{a} > 0]\!] &= [\![\overline{a} = 1]\!] = H_A^T(a); \\ [\![\overline{a} = 0]\!] &= \{\alpha \in X : a \in \text{supp}(\alpha)\}. \end{cases}$$

The first two sets in (*) are quasi-compact open in Y_T (spectral topology), while the third is *closed* in Y_T. ■

PROPOSITION 6.9. *Let* $\langle A, T\rangle$ *be a proper p-ring, let* $Y_T = \text{Sper}(A, T)$ *and let* $n \geq 1$ *be an integer. With notation as in* 4.10*, set* $\mathfrak{r}_{A,T} : G_T(A) \longrightarrow \mathscr{G}^*_{A,T}$*, defined by* $\mathfrak{r}_{A,T}(a^T) = \overline{a}$*. Then,*

a) $\mathfrak{r} = \mathfrak{r}_{A,T}$ *is a group morphism, taking* -1 *to* -1 *and for all n-forms* φ *over* $A^\times$*,* $\mathfrak{r}[D_T(\varphi^T)] \subseteq \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{\varphi})$*, where* $D_T(\cdot)$ *means representation in* $G_T(A)$ (*cf.* 2.25)*, while* $\mathscr{D}_{\mathscr{G}_{A,T}}(\cdot)$ *denotes the* (*ordinary*) *value set in the real semigroup* $\mathscr{G}_{A,T}$*. Moreover,*

$(*)$ $\quad\mathfrak{r}$ *is injective* $\Leftrightarrow$ *The preorder* T *is unit-reflecting* (*cf.* 8.6).

b) If A is a ring with many units, then $\mathfrak{r}$ *is a group isomorphism and for* $a, b_1, \ldots, b_n \in A^\times$

$$(**)\qquad \overline{a} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{b}_1, \ldots, \overline{b}_n) \quad\Leftrightarrow\quad a \in D^v_T(b_1, \ldots, b_n).$$

In particular, if A verifies the hypotheses in the statement of Theorem 6.5*, then for all preorders* T *of* A*,* $\mathscr{G}^*_{A,T}$ *is a reduced special group, isomorphic to* $G_T(A)$.

PROOF. a) Obviously $\overline{x}$ is a unit in $\mathscr{G}_{A,T}$ for any $x \in A^\times$. To show that $\mathfrak{r}$ is well-defined, let $a^T = c^T$; then $ac \in T^\times$, and therefore $ac \in \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha)$. We conclude that $\pi_\alpha(a) >_\alpha 0$ iff $\pi_\alpha(c) >_\alpha 0$, for $\alpha \in Y_T$, which immediately implies $\overline{a} = \overline{c}$. Clearly, $\mathfrak{r}(1) = 1$ and $\mathfrak{r}(-1) = -1$; moreover, if $a, c \in A^\times$, then

$$\mathfrak{r}(a^T \cdot c^T) = \mathfrak{r}((ac)^T) = \overline{ac} = \overline{a} \cdot \overline{c},$$

and $\mathfrak{r}$ is a group morphism. Let φ be a n-form over $A^\times$ and assume that $a^T \in D_T(\varphi^T)$. Then (cf. 2.25), there are $x_2, \ldots, x_n \in A^\times$ such that $\langle a^T, x_2^T, \ldots, x_n^T\rangle \equiv_T \varphi^T$. By Corollary 2.22, this isometry yields $\langle a, x_2, \ldots, x_n\rangle \approx_T \varphi$, and Lemma 3.4 implies $a \in D^v_T(\varphi)$; so Definition 6.8.(i) entails $\overline{a} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{\varphi})$, as needed.

In order to prove the implication ($\Rightarrow$) in (*), it is enough to check that $A^\times \cap \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha) \subseteq T^\times$ (cf. 8.5 (1) and 8.6). If $a \in A^\times$ is strictly positive on all orderings in Y_T, then $\overline{a} = 1$ and the injectivity of $\mathfrak{r}$ entails $a^T = 1$, i.e., $a \in T^\times$. Conversely, if T is unit-reflecting and $a \in A^\times$ is such that $\overline{a} = 1$, then $a \in A^\times \cap \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha) = T^\times$, and so $a^T = 1$, showing that $\ker \mathfrak{r} = \{1\}$, as needed.

b) By Proposition 8.1.4, p. 153, of [**Mar**], every unit in $\mathscr{G}_{A,T}$ is of the form $\overline{a}$, for some $a \in A^\times$. It is then immediate that $\mathfrak{r}$ is a surjection. Since every preorder of A is unit-reflecting (Corollary 4.1.9, p. 33, of [**Wa**]), it follows from (a) that $\mathfrak{r}$ is a bijection. Implication ($\Leftarrow$) in (**) is immediate from Definition 6.8.(i). For the converse, assume that $\overline{a} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{b}_1, \ldots, \overline{b}_n)$, with $a, b_1, \ldots, b_n \in A^\times$. Scaling, if necessary, we may assume that $b_1 = 1$. Hence (6.8.(i)), there are $t, w_1, \ldots, w_n \in T$ such that

$$\text{(I)}\qquad ta = w_1 + w_2 b_2 + \ldots + w_n b_n, \quad \text{with} \quad \overline{t}\,\overline{a} = \overline{a}.$$

Since $\overline{a}$ is a unit in $\mathscr{G}_{A,T}$, we get $\overline{a}^2 = 1$, and the second equation in (I) yields $\overline{t} = \overline{t}\,\overline{a}^2 = \overline{a}^2 = 1$. By Theorem 4.11.(d), there are an integer $k \geq 0$ and s, $t_1 \in T$ such that

$$st \;=\; (1 + t^2)^k + t_1 \;=\; 1 + t',$$

with $t' \in T$. Scaling (I) by s we may assume that

(II) $$\text{For all } \alpha \in Y_T, \quad \pi_\alpha(t) \geq_\alpha 1.$$

We claim that the polynomial $Q(Y) = 1 + a^2 + Y^2$ has local unit values in A. Indeed, let $\mathfrak{m}$ be a maximal ideal in A; if $1 + a^2 \notin \mathfrak{m}$, then $Q(0) \notin \mathfrak{m}$; if $1 + a^2 \in \mathfrak{m}$, then $Q(1) \notin \mathfrak{m}$, otherwise $1 \in \mathfrak{m}$, which is impossible. Since A has many units, there is $c \in A$ such that $Q(c) = 1 + a^2 + c^2 \in A^\times$, and so, is an element of $T^\times$. Since for each z in a linearly ordered ring we have $|z| \leq \dfrac{1 + z^2}{2} < 1 + z^2$, it follows that if $s' = \dfrac{1}{1 + a^2 + c^2}$, then $s' \in T^\times$, $s'a \in A^\times$ and $|\pi_\alpha(s'a)| <_\alpha 1$, for all $\alpha \in Y_T$. Thus, (II) entails

(III) $$\text{For all } \alpha \in Y_T, \quad \pi_\alpha(t + s'a) >_\alpha 0.$$

Next, observe that the polynomial $P(X) = t(1 + X^2) + s'aX^2$ has local unit values in A. Indeed, if $\mathfrak{m}$ is a maximal ideal in A, there are two possibilities:

- If $t \notin \mathfrak{m}$, then $P(0) = t \notin \mathfrak{m}$;
- If $t \in \mathfrak{m}$, then $P(1) \notin \mathfrak{m}$, otherwise the unit $s'a$ would be in $\mathfrak{m}$.

Hence, there is $d \in A$ such that

$P(d) \;=\; t(1 + d^2) + s'ad^2 \;=\; t + (t + s'a)d^2 \in A^\times.$

Now (II) and (III) yield $\pi_\alpha(P(d)) >_\alpha 0$, for every $\alpha \in Y_T$, that is,

$P(d) \in A^\times \cap \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha).$

Since T is unit-reflecting, we obtain $P(d) \in T^\times$. But then (I) yields

$$\begin{aligned} P(d)a &= [t(1 + d^2) + s'ad^2]a \;=\; ta(1 + d^2) + s'a^2d^2 \\ &= (1 + d^2)(w_1 + w_2b_2 + \ldots + w_nb_n) + s'a^2d^2 \\ &= [(1 + d^2)w_1 + s'a^2d^2] + (1 + d^2)w_2b_2 + \ldots + (1 + d^2)w_nb_n, \end{aligned}$$

which entails $a \in D^v_T(1, b_2, \ldots, b_n)$, because $P(d) \in T^\times$, as needed.

If A satisfies the hypotheses in the statement of 6.5, then for all preorders T of A, $\langle A, T \rangle$ is T-faithfully quadratic; thus, $G_T(A)$ is a reduced special group and for all $a, b_1, \ldots, b_n \in A^\times$,

$$a \in D^v_T(b_1, \ldots, b_n) \quad \text{iff} \quad a^T \in D(b_1^T, \ldots, b_n^T) \text{ in } G_T(A).$$

Hence, the isomorphism $\mathfrak{r}$ and (**) guarantee that $\mathscr{G}^*_{A,T}$ is a reduced special group, as claimed. ■

3. Rings of Formal Power Series

New examples of rings with many units (to the best of the authors' knowledge), come from power series rings, with coefficients in a ring with many units.

6.10. Notation Let A be a commutative and let $R = A[[X]]$ be the ring of formal power series with coefficients in A. For $S \in R$, we have

(I) $$S \;=\; \sum_{i=0}^{\infty} S_i X^i \;=\; S_0 + S(X),$$

where S_0 is the *constant term* of S and $S(X)$ is in the ideal XR. ■

We register the following:

FACT 6.11. *Let A be a commutative ring and let $R = A[[X]]$.*

a) [Proposition 1.2, p. 1, [**Ben**]] *An element of R is a unit in R iff its constant term is a unit in A.*

b) [Corollaire 1.6, p. 4, [**Ben**]] *An ideal I of R is maximal iff it is of the form $\mathfrak{m} + XR$, where $\mathfrak{m}$ is a maximal ideal of A.*

c) For $P(Y_1, \ldots, Y_n) \in R[Y_1, \ldots, Y_n]$, let $P_c(Y_1, \ldots, Y_n) \in A[Y_1, \ldots, Y_n]$ be the polynomial whose coefficients are the constant terms of the coefficients of P. Then, for all $S_1, \ldots, S_n \in R$,

$$P(S_1, \ldots, S_n) \;=\; P_c(S_{10}, \ldots, S_{n0}) + \textit{terms in } XR.$$

PROOF. We comment only on (c); with notation as in (I) of 6.10, if $M_\nu(Y_1, \ldots, Y_n) = a_\nu Y_1^{\nu_1} \cdots Y_n^{\nu_n}$ is a monomial in P and $S_1, \ldots, S_n \in R$, then

$$M_\nu(S_1, \ldots, S_n) = (a_{\nu 0} + a_\nu(X))(S_{10} + S_1(X))^{\nu_1} \cdots (S_{n0} + S_n(X))^{\nu_n}.$$

Clearly, we may rewrite the preceding equality as

$$M_\nu \;=\; a_{\nu 0} S_{10}^{\nu_1} \cdots S_{n0}^{\nu_n} + \text{terms in } XR,$$

so that, by adding all monomials in P, we obtain $P(S_1, \ldots, S_n) = P_c(S_{10}, \ldots, S_{n0}) +$ terms in XR, as needed. ■

PROPOSITION 6.12. *If A is a ring with many units, then $A[[X]]$ has also many units.*

PROOF. Notation as in 6.10 and 6.11.(c), we show that if $P(Y_1, \ldots, Y_n)$ is a polynomial with coefficients in $R := A[[X]]$, then

(*) $$\begin{array}{c} P(Y_1, \ldots, Y_n) \text{ has local unit values in } R \\ \Updownarrow \\ P_c(Y_1, \ldots, Y_n) \text{ has local unit values in } A. \end{array}$$

($\Downarrow$). Assume that P has local unit values in R and let $\mathfrak{m}$ be a maximal ideal in A. By Fact 6.11.(b), $\mathfrak{m} + XR$ is a maximal ideal in R, and so there are $S_1, \ldots, S_n \in R$ such that $P(S_1, \ldots, S_n) \notin \mathfrak{m} + XR$. By item (c) in Fact 6.11, we have

$$P(S_1, \ldots, S_n) \;=\; P_c(S_{10}, \ldots, S_{n0}) + \text{terms in } XR,$$

wherefrom we conclude that $P_c(S_{10}, \ldots, S_{n0}) \notin \mathfrak{m}$, as needed.

($\Uparrow$). If M is a maximal ideal in R, by 6.11.(b), $M = \mathfrak{m} + XR$, for some maximal ideal $\mathfrak{m}$ in A. Since $P_c(Y_1, \ldots, Y_n)$ has local unit values in A, there are $a_1, \ldots, a_n \in A$ such that $P_c(a_1, \ldots, a_n) \notin \mathfrak{m}$. But then, it is clear that $P(a_1, \ldots, a_n) = P_c(a_1, \ldots, a_n) +$ terms in XR, cannot be in M.

To finish the proof, suppose $P(Y_1, \ldots, Y_n)$ has local unit values in R; by (*), $P_c(Y_1, \ldots, Y_n)$ has local unit values in A, and so there are $c_1, \ldots, c_n \in A$ so that $P_c(c_1, \ldots, c_n) \in A^\times$. Since $P(c_1, \ldots, c_n) = P_c(c_1, \ldots, c_n) +$ terms in XR (6.11.(c)), it follows from item (a) in 6.11 that $P(a_1, \ldots, a_n) \in R^\times$, as needed. ■

COROLLARY 6.13. *If A is a ring with many units, then*

a) For all $n \geq 1$, $A[[X_1, \ldots, X_n]]$ is a ring with many units.

b) The ring of formal power series with coefficients in A, in any number of variables, is a ring with many units.

PROOF. Item (a) follows immediately from 6.12 and induction, while (b) is a consequence of Corollary 6.4 and the fact that the ring of formal power series in any number of variables is the inductive limit of rings of power series in a finite number of variables. ■

CHAPTER 7

Transversality of Representation in p-rings with Bounded Inversion

The aim of this Chapter is to prove that value representation in bounded inversion rings (BIRs) is transversal, i.e., satisfies the analog of axiom [T-FQ 1] for forms of arbitrary dimension. Our proof is couched in the language of real semigroups (cf. 6.7). We also introduce a notion of "BIR hull" of a p-ring and prove some of its basic properties (cf. 7.4). Notation is as in 4.10.

1. Rings with Bounded Inversion

DEFINITION 7.1. *Let $\langle A, T \rangle$ be a p-ring.*

a) $\langle A, T \rangle$ has the **bounded inversion property** *(or is a* **bounded inversion (preordered) ring***, abbreviated* **BIR***) if*

$$1 + T \ = \ \{1 + t : t \in T\} \subseteq A^{\times}.$$

b) In case $T = \Sigma A^2$, A is said to have the **weak bounded inversion property**.[1]

■

These notions are standard in real algebra (cf. e.g., [**SM**], [**KZ**]). It is known from the literature that the BIR property has several equivalent formulations (cf. Proposition 7.1, p. 55, [**KZ**]). For the reader's convenience, we sketch a proof.

PROPOSITION 7.2. *If $\langle A, T \rangle$ is a p-ring and $Y_T = \mathrm{Sper}(A, T)$, the following are equivalent:*

(1) *Every maximal ideal of A is T-convex* (4.14.(*a*))*;*

(2) *$\langle A, T \rangle$ is a BIR;*

(3) $\bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha) \ = \ T^{\times}$.

PROOF. (1) ⇒ (2): If every maximal ideal of A is T-convex but $1 + t \notin A^{\times}$, there is a maximal ideal, $\mathfrak{m}$, such that $1 + t \in \mathfrak{m}$. The T-convexity of $\mathfrak{m}$ implies, then, $1 \in \mathfrak{m}$, a contradiction.

(2) ⇒ (1) : Since maximal ideals are radical, if some maximal ideal $\mathfrak{m}$ of A fails to be T-convex, it will not be T-radical (cf. 4.14.(a)) and so there are $a \in A$, $t \in T$ so that $a^2 + t \in \mathfrak{m}$, but $a \notin \mathfrak{m}$ (whence $t \notin \mathfrak{m}$). Then, in the field $A/\mathfrak{m}$, we have $a^2/\mathfrak{m} = -t/\mathfrak{m} \neq 0$. Let $b/\mathfrak{m}$ be the inverse of $a/\mathfrak{m}$ in $A/\mathfrak{m}$. Multiplying the equality $a^2/\mathfrak{m} = -t/\mathfrak{m}$ by $b^2/\mathfrak{m}$ gives $(ab)^2/\mathfrak{m} = 1 = -tb^2/\mathfrak{m}$, that is, $1 + b^2 t \in \mathfrak{m}$, contradicting the fact that $\langle A, T \rangle$ is a BIR.

[1] Also called a **regular function ring** (RFR), see [**Ma1**], Def. 4.1, p. 615.

(2) $\Rightarrow$ (3) : Since units cannot belong to proper prime ideals and $T \subseteq \alpha$ for every $\alpha \in Y_T$, we immediately conclude $T^\times \subseteq \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha)$. For the reverse inclusion, if $x \notin T^\times$ we shall construct $\alpha \in Y_T$ such that $x \notin \alpha \setminus (-\alpha)$. Set $T' = T - xT$; T' is a preorder. We show:

• T' is proper. Otherwise, $-1 \in T'$, i.e., $-1 = s - xt$, with $s, t \in T$, or equivalently, $xt = 1 + s$. Since $\langle A, T \rangle$ is a BIR, $1 + s \in T^\times$, whence xt, with $t \in T^\times$. It follows that $x = (1+s)/t \in T^\times$, a contradiction.

By Proposition 4.14.2.(1), A has a proper T'-convex ideal; let I be a maximal one. By 4.14.2.(2), I is prime, and by Proposition 4.14.1 there is $\alpha \in \mathrm{Sper}(A)$ such that $T' \subseteq \alpha$ and $I = \mathrm{supp}(\alpha)$. Then, $-x \in T' \subseteq \alpha$, i.e., $-\pi_\alpha(x) \geq_\alpha 0$, which entails $x \notin \alpha \setminus (-\alpha)$, as needed.

(3) $\Rightarrow$ (2) : For all $\alpha \in Y_T$ and $t \in T$, $\pi_\alpha(1+t) = 1 + \pi_\alpha(t) >_\alpha 0$, i.e., $1 + t$ is strictly positive in all orderings of A containing T. Hence, (3) entails $1 + t \in T^\times \subseteq A^\times$ and $\langle A, T \rangle$ is a BIR. ■

Remark 7.3. Proposition 7.2 shows that

$$T^\times = A^\times \cap \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha)$$

holds in every BIR. However, it holds in other rings as well: for example, in rings with many units where 2 is invertible; see [**Wa**], Cor. 4.1.9, p. 33. This type of preorder, called *unit-reflecting* (cf. Definition 8.6), will play an important role in Chapters 8 and 9. ■

To a p-ring $\langle A, T \rangle$ we can associate a "BIR hull", which in interesting cases is an extension of $\langle A, T \rangle$; moreover, if $\langle A, T \rangle$ is a BIR there is a BIR, $\langle R, P \rangle$ which is P-reduced (cf. Definition 4.14.4) and the reduced π-special groups $G_T(A)$ and $G_P(R)$ are *isomorphic.*

Proposition 7.4. *Let $\langle A, T \rangle$ be a p-ring.*

a) [2] *The set $S = 1 + T$ is a proper multiplicative subset of A. Moreover, if $\nu : A \longrightarrow A^* = AS^{-1}$ is the ring of fractions of A by S and*

$$T^* = \{t/s^2 \in A^* : t \in T \text{ and } s \in S\},$$

then

(1) *T^* is a proper preorder of A^* and $\langle A^*, T^* \rangle$ is a BIR.*

(2) *ν is a p-ring morphism; moreover, if A is T-reduced* (cf. 4.14.(a))*, then ν is injective.*

(3) *If $f : \langle A, T \rangle \longrightarrow \langle R, P \rangle$ is a p-ring morphism and $\langle R, P \rangle$ has bounded inversion, there is a* unique *p-ring morphism, $g : \langle A^*, T^* \rangle \longrightarrow \langle R, P \rangle$, such that $g \circ \nu = f$.*

b) If $\langle A, T \rangle$ is a BIR, let $R = A/\sqrt[T]{0}$, $P = T/\sqrt[T]{0}$. Then, $\langle R, P \rangle$ is a BIR and the canonical p-ring morphism, $\gamma : \langle A, T \rangle \longrightarrow \langle R, P \rangle$, induces an **isomorphism** *of π-RSGs, $\gamma^\pi : G_T(A) \longrightarrow G_P(R)$.*

Proof. a) We prove only item (2). The first assertion is clear. For the second, suppose $\nu(a) = \dfrac{a}{1} = 0$, with $a \in A$. Then, there is $s \in S$ such that $sa = 0$. Since

[2] It was pointed out by the referee that this is a known result.

$s = 1 + t$, $t \in T$, we have: $0 = sa^2 = (1+t)a^2 = a^2 + ta^2$. Since $ta^2 \in T$, we obtain $a \in \sqrt[T]{0} = (0)$, whence $a = 0$.

b) We do the proof in several steps.

(1) $\langle R, P \rangle$ is a proper p-ring.

Proof of (1). Clearly, $2 \in R^\times$, P is closed under products, sums, and contains R^2; so it is a preorder of R. To prove P is proper, assume, towards a contradiction, that $-1/\sqrt[T]{0} = t/\sqrt[T]{0}$ for some $t \in T$. Then, $1 + t \in \sqrt[T]{0}$ and (by Proposition 4.14.3) there are $k \geq 0$ and $s \in T$ such that $(1+t)^{2k} + s = 0$. If $k = 0$, we get $1 + s = 0$ and so $-1 \in T$, a contradiction. Similarly, if $k > 0$, then $(1+t)^{2k} + s = 1 + t' + s = 0$, with $t' \in T$, which also implies that T is not proper. Note that the BIR property of $\langle A, T \rangle$ is not used to obtain (1).

(2) Clearly, the natural quotient morphism, $\gamma : A \longrightarrow R$ is a p-ring morphism; in fact, $P = \gamma[T]$.

(3) R is P-reduced.

Proof of (3). Assume $\gamma(a) \in \sqrt[P]{0}$, with $a \in A$. Then, there are $k \geq 0$ and $t \in T$ so that

$$\gamma(a)^{2k} + \gamma(t) = \gamma(a^{2k} + t) = 0. \qquad \text{(in } R\text{)}$$

i.e. $a^{2k} + t \in \ker \gamma = \sqrt[T]{0}$. Then, there are an integer $l \geq 0$ and $s \in T$ such that

$$(*) \qquad (a^{2k} + t)^{2l} + s = 0.$$

Since a^{2k}, $t \in T$, we can write $(a^{2k} + t)^{2l} = a^{4kl} + t'$, with $t' \in T$, and (*) yields $a^{4kl} + t' + s = 0$, whence $a \in \sqrt[T]{0}$. It follows that $\gamma(a) = 0$, proving (3).

(4) $\gamma^{-1}[1] \subseteq T^\times$.

Proof of (4). Let $c \in A$ be such that $\gamma(c) = 1$. Then, $1 - c \in \ker \gamma = \sqrt[T]{0}$, i.e., there are $k \geq 0$ and $t \in T$ so that $(1-c)^{2k} + t = 0$. Since $\langle A, T \rangle$ is a BIR, $k > 0$. Separating even and odd powers in the binomial development of $(1-c)^{2k}$, we can write

$$(1-c)^{2k} = 1 - cr + r',$$

with $r, r' \in \Sigma A^2 \subseteq T$. Thus, $cr = 1 + r' + t \in T^\times \subseteq A^\times$. It follows that $c, r \in A^\times$ and, since $r \in T^\times$, we also get $c = cr/r \in T^\times$, as claimed.

Recall that the domain of $G_T(A)$ is $A^\times/T^\times = \{a^T : a \in A^\times\}$ (similarly for $G_P(R)$). By 2.8, the map $\gamma^\pi : G_T(A) \longrightarrow G_P(R)$, given by $\gamma^\pi(a^T) = \gamma(a)^P$, is a π-SG morphism. As a corollary of (4) we have:

(5) γ^π is injective.

Proof of (5). Let $a \in A^\times$ be such that $\gamma^\pi(a^T) = \gamma(a)^P = 1$. This means that $\gamma(a) \in P^\times = T^\times/\sqrt[T]{0}$, i.e., $\gamma(a) = a/\sqrt[T]{0} = t/\sqrt[T]{0}$, for some $t \in T$. Thus, $\gamma(a - t) = 0$ and (since $a \in A^\times$), $\gamma(t/a) = 1$. By (4), $t/a \in T^\times$, which implies $a = t/t'$, for some $t' \in T^\times$, whence $a^T = 1$.

(6) γ^π is surjective.

Proof of (6). Let $u \in R^\times$ and select $a \in A$ such that $\gamma(a) = u$. Then, there is $b \in A$ so that $\gamma(ab) = \gamma(a)\gamma(b) = 1$. By (4), $ab \in T^\times$, which in turn implies $a \in A^\times$. Hence, we get $\gamma^\pi(a^T) = \gamma(a)^P = u^P$.

(7) For $a,\, b \in A^\times$, $\gamma(a)^P \in D_P(1,\, \gamma(b)^P) \;\Rightarrow\; a^T \in D_T(1,\, b^T)$.

Proof of (7). By 2.7.(a.1), the assumption entails

$$\gamma(a) \in D_P^v(\gamma(1),\, \gamma(b)), \text{ i.e., } \gamma(a) = u\gamma(1) + v\gamma(b),$$

for some $u,\, v \in P$. Since $P = \gamma[T]$, there are $x,\, y \in T$ so that $u = \gamma(x)$ and $v = \gamma(y)$, whence $\gamma(a) = \gamma(x + yb)$. From $a \in A^\times$, we get $\gamma\left(\frac{x+yb}{a}\right) = 1$. By (4), $\frac{x+yb}{a} = t \in T^\times$, and so $at = x + yb$. Since $x,\, y \in T$, we get $a \in D_T^v(1,\, b)$, whence (by 2.7.(a.1)), $a^T \in D_T(1,\, b^T)$, ending the proof. ■

Remark 7.5. Let $\langle A, T \rangle$ be a BIR and let $\langle R, P \rangle$ be as in the statement of Proposition 7.4.(b). Since supp(β) is a T-convex prime ideal for all $\beta \in \mathrm{Sper}(A, T)$ and $\sqrt[T]{0}$ is the intersection of all T-convex primes in A, it follows from known results on the real spectra of quotients that the dual of the quotient map $\gamma : \langle A, T \rangle \longrightarrow \langle R, P \rangle$, namely,

$$\mathrm{Sper}(\gamma) : \mathrm{Sper}(R, P) \longrightarrow \mathrm{Sper}(A, T),$$

is a homeomorphism. Therefore, we have both an isomorphism of the corresponding π-RSGs (7.4.(b)), and a homeomorphism of the associated real spectra. ■

2. Transversality of Value Representation

The aim of this section is to prove that value representation mod T by arbitrary (diagonal) quadratic forms with unit coefficients over a BIR, $\langle A, T \rangle$, is transversal; in other words, $\langle A, T \rangle$ verifies axiom [T-FQ 1], *for forms of arbitrary dimension.* For the case of weak bounded inversion (i.e., $T = \Sigma A^2$), this result appears in [**Ma2**], Theorem 3.4. Here we give a shorter and more structural proof that applies to arbitrary BIRs. In terms of Definition 2.24, the result to be proved is item (d) of the following (for notation see 4.14 and 6.7):

Theorem 7.6. *Let $\langle A, T \rangle$ be a BIR. Then:*

a) For $a \in A$, $\overline{a}$ is invertible in $\mathscr{G}_{A,T}$ $\;\Leftrightarrow\;$ $a \in A^\times$.

b) For $a \in A^\times$ and $a' \in A$, $\overline{a} = \overline{a'}$ $\;\Leftrightarrow\;$ $\exists\, r \in T^\times$ so that $a = ra'$.

c) For $b,\, a_1, \ldots, a_n \in A^\times$, $\overline{b} \in \mathscr{D}^t_{\mathscr{G}_{A,T}}(\overline{a}_1, \ldots, \overline{a}_n)$ $\;\Leftrightarrow\;$ $b \in D_T^t(a_1, \ldots, a_n)$.

d) For all $a_1, \ldots, a_n \in A^\times$, $D_T^v(a_1, \ldots, a_n) \;=\; D_T^t(a_1, \ldots, a_n)$.

Proof. a) The implication ($\Leftarrow$) is clear. For the converse, assume that $\overline{a}$ is a unit in $\mathscr{G}_{A,T}$, i.e., it only takes on the values ± 1. Hence, $\overline{a}^2 = 1$ and so by 7.2, $a^2 \in \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha) = T^\times$, whence $a \in A^\times$.

b) Implication ($\Leftarrow$) is clear since $r \in T^\times$ entails $\overline{r}(\alpha) = 1$ for all $\alpha \in \mathrm{Sper}(A,\, T)$ (cf. Proposition 7.2.(3)). For the converse, $\overline{a} = \overline{a'}$ entails $\overline{aa'} = 1$ and, by Proposition 7.2, $aa' \in T^\times$; take $r = \frac{aa'}{a'^2} = \frac{a}{a'}$.

c) Fix $b \in A^\times$. If $b \in D_T^t(a_1, \ldots, a_n)$ (see Definition 2.24 above), there are $c_1, \ldots, c_n \in T^\times$ such that $b = \sum_{i=1}^n c_i a_i$; since $\overline{a}_i \overline{c}_i = \overline{a}_i$, $1 \le i \le n$, it follows immediately from the definition of transversal value set in 6.8(ii) that $\overline{b} \in \mathscr{D}^t_{\mathscr{G}_{A,T}}(\overline{a}_1, \ldots, \overline{a}_n)$. Conversely, if $\overline{b} \in \mathscr{D}^t_{\mathscr{G}_{A,T}}(\overline{a}_1, \ldots, \overline{a}_n)$, let $b',\, a'_1, \ldots, a'_n \in A$ be such that $\overline{b} = \overline{b'}$,

$\overline{a_i} = \overline{a_i'}$ $(1 \leq i \leq n)$ and $b' = \sum_{i=1}^n a_i'$. By (b), there are $r, s_1, \ldots, s_n \in T^\times$ such that $b = b'r$ and $a_i' = a_i s_i$; hence, $b = rb' = \sum_{i=1}^n (rs_i)a_i$, with $rs_i \in T^\times$, $1 \leq i \leq n$, and $b \in D_T^t(a_1, \ldots, a_n)$, as desired.

d) Only the inclusion $\subseteq$ needs proof. Let $b \in D_T^v(a_1, \ldots, a_n)$, with $b, a_1, \ldots, a_n \in A^\times$; then, it is clear from Definition 6.8.(i) that $\overline{b} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{a}_1, \ldots, \overline{a}_n)$. Now, we use the following law, valid in any real semigroup G; cf. Proposition I.2.7 (3) in [**DP2**] and the Remark below: for *arbitrary* $x, y_1, \ldots, y_n \in G$

$$(*) \qquad x \in \mathscr{D}_{\mathscr{G}}(y_1, \ldots, y_n) \;\Rightarrow\; x \in \mathscr{D}^t_{\mathscr{G}}(y_1x^2, \ldots, y_nx^2).$$

This yields: $\overline{b} \in \mathscr{D}^t_{\mathscr{G}_{A,T}}(\overline{a_1b^2}, \ldots, \overline{a_nb^2})$. Since $b \in A^\times$, item (a) guarantees that $\overline{b}$ is a unit in $\mathscr{G}_{A,T}$, whence $\overline{b}^2 = 1$. Thus, $\overline{b} \in \mathscr{D}^t_{\mathscr{G}_{A,T}}(\overline{a}_1, \ldots, \overline{a}_n)$ and the conclusion follows from (c). ■

Remark. A direct proof of the law (*) for abstract real spectra of type $\langle \mathrm{Sper}(A,T), G_{A,T} \rangle$ is given in [**Mar**], Proposition 5.5.2, pp. 96-97. In the dual category of real semigroups, of [**DP1**], [**DP2**], (*) is an axiom for $n = 2$ (axiom [RS 6], Def. 2.1 of [**DP1**]), and is proven by induction for $n \geq 3$; cf. [**DP1**], Prop, 2.7 (3), pp. 110–112. ■

Example 7.7. The transversality of Theorem 7.6.(d) *is not* valid if representation modulo a preorder T is replaced by representation modulo squares, even for rings with weak bounded inversion. An example is given in [**Ma2**], which we reproduce here for the reader's benefit.

Let $A = \mathbb{R}[X]S^{-1}$, where $S = \{f \in \mathbb{R}[X] : f > 0 \text{ on all of } \mathbb{R}\}$. It is easily checked that A has weak bounded inversion and, obviously, $1 + X^2 \in D^v(1, 1)$; but $1 + X^2 \notin D^t(1, 1)$; otherwise, we would have $1 + X^2 = (f_1/g)^2 + (f_2/g)^2$, with $f_1, f_2, g > 0$ on all of $\mathbb{R}$. Then, these polynomials are of even positive degree and their square classes have degree $\equiv 0 \pmod 4$; the identity $g^2(1 + X^2) = f_1^2 + f_2^2$ is then impossible in $\mathbb{R}[X]$, since each side has a different degree mod 4. ■

Remark. Preordered rings with many units satisfy [Σ-FQ 1], but may not have weak bounded inversion. For example, let F be any field with at least 7 elements (cf. 6.5). The ring $\mathbb{R} \times F$ has many units, is preordered by $\mathbb{R}^+ \times F$, but is not of weak bounded inversion since $\langle 1, -1 \rangle \in \mathbb{R}^+ \times F$, but $\langle 1, 1 \rangle + \langle 1, -1 \rangle$ is not a unit. ■

As in the case of rings with many units (cf. 6.9), there is a close relation between the π-SG of a p-ring $\langle A, T \rangle$ with bounded inversion and the group of units of the real semigroup associated to $\langle A, T \rangle$. Moreover, if A is T-faithfully quadratic, then $\mathscr{G}^*_{A,T}$ is a reduced special group, isomorphic to $G_T(A)$.

Proposition 7.8. *Let $\langle A, T \rangle$ be a p-ring with bounded inversion. With notation as in* 6.9*, the map $\mathfrak{r} : G_T(A) \longrightarrow \mathscr{G}^*_{A,T}$ is a group isomorphism sending -1 to -1 and such that for all $a, b \in A^\times$,*

$$(\dagger) \qquad \overline{a} \in \mathscr{D}_{\mathscr{G}_{A,T}}(\overline{1}, \overline{b}) \;\Leftrightarrow\; a \in D_T^v(1, b) \;\Leftrightarrow\; a^T \in D(1, b^T) \text{ in } G_T(A).$$

Moreover, if $\langle A, T \rangle$ is faithfully quadratic, then $\mathscr{G}^*_{A,T}$, with binary representation induced by that of $\mathscr{G}_{A,T}$, is a reduced special group.

PROOF. By Propositions 6.9.(a) and 7.2, $\mathfrak{r}$ is injective, while Theorem 7.6.(a) guarantees it is surjective. The second equivalence in (†) follows from 2.7.(a.1), while the implication ($\Leftarrow$) in the first equivalence is clear. For the implication ($\Rightarrow$) in the first equivalence, assume, for a, $b \in A^{\times}$, that $\overline{a} \in \mathscr{D}_{\mathscr{G}_{A,T}}(1, \overline{b})$. By Definition 6.8.(i), there are t, t_1, t_2 in T such that $ta = t_1 + t_2 b$ and $\overline{at} = \overline{a}$. Since $\overline{a} \in \mathscr{G}^*_{A,T}$, we have $\overline{a}^2 = 1$, and so $\overline{t} = 1$. Hence, $t \in \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha) = T^{\times}$ (by Proposition 7.2). Therefore, $a = (t_1/t) + (t_2/t)b$, whence $a \in D^v_T(1, b)$, as needed. It follows from (†) that $\mathfrak{r}$ is an isomorphism of π-SGs. In case $\langle A, T \rangle$ is T-faithfully quadratic, $G_T(A)$ is a reduced special group, and the same will be true of $\mathscr{G}^*_{A,T}$. ■

CHAPTER 8

Reduced f-Rings

In this Chapter we discuss the quadratic faithfulness of the rings in the title. **In all that follows, the expression "f-ring" will stand for "*reduced* f-ring."**, i.e., a f-ring without non-zero nilpotent elements.

Section 1 collects, for later use and ready reference, a few basic facts about lattice-ordered rings and f-rings, and introduces the notion of *unit-reflecting* preorder that will be important in the sequel.

Section 2 is mainly devoted to the description of an orthogonal decomposition of a f-ring associated to each of its units (8.13), of constant use in this Chapter.

Our main result in section 3 is Theorem 8.21, which establishes the T-quadratic faithfulness of all p-rings $\langle A, T \rangle$, where A is a (reduced) f-ring and T is a preorder containing the canonical partial order $T^A_\sharp$ of A. Further, we prove that the reduced special group, $G_T(A)$, associated to $\langle A, T \rangle$ is a Boolean algebra. This is accomplished in two steps: first, we establish the result in the case $T = T^A_\sharp$ (Theorem 8.20), and then prove it for arbitrary $T \supseteq T^A_\sharp$, by use of Theorem 3.9, a result allowing considerable simplification in the arguments. Among other things, in the important case of rings of real-valued continuous functions and bounded continuous functions on a topological space X, Theorem 8.21 yields *complete* quadratic faithfulness of all rings $\mathbb{C}$(X) and $\mathbb{C}_b(X)$.

Whenever X is, in addition, compact and Hausdorff, we prove, using partitions of unity, that *all* preorders on $\mathbb{C}(X)$ are unit-reflecting (Theorem 8.29). This is done in section 4. The Chapter closes with some applications and examples of the above results.

Remark. Some of the proofs given in this Chapter can be rendered by representing the (reduced) f-rings involved as subdirect products of totally ordered domains. This procedure is analogous to the use of "coordinates" in Linear Algebra and Geometry. However, *with possible generalizations in mind*, we have preferred, whenever feasible, an "intrinsic" approach, which, in our context, consists in casting proofs in terms of the order and lattice structure of the f-rings at hand. Another instance of this "intrinsic" approach is [**KZ**]. ■

1. Preliminaries

8.1. Lattice-Ordered Rings and f-rings.

Here we collect, for ready reference, a few basic facts about lattice-ordered rings and f-rings used in the text. Our basic reference for lattice-ordered rings and f-rings is [**BKW**], particularly Chapters 8 and 9. We note that, when dealing with lattice-ordered groups, [**BKW**] uses multiplicative notation. The reader may also consult [**S**].

1. Lattice-Ordered Rings. A po-ring $\langle A, \leq \rangle$ is **lattice-ordered (ℓ-ring)** if for all a, $b \in A$,

$$a \vee b = \sup\{a, b\} \quad \text{and} \quad a \wedge b = \inf\{a, b\}$$

exist in A, where join (or sup) and meet (or inf) are considered with respect to the partial order $\leq$. Let A be a ℓ-ring and let $a \in A$. Define

(av) $$a^+ = a \vee 0, \quad a^- = -a \vee 0 \quad \text{and} \quad |a| = a^+ \vee a^-,$$

called the **positive part, negative part** and **absolute value of a in A**. It is clear that a^+, a^-, $|a| \geq 0$. We note the following

FACT 8.2. ([**BKW**], 8.1.4, p. 151) *If A is an ℓ-group*[1] *and a, b, $x \in A$, then*

a) $x + (a \wedge b) = (x + a) \wedge (x + b)$ *and*

$x + (a \vee b) = (x + a) \vee (x + b)$.

b) $-(a \wedge b) = -a \vee -b$ *and* $-(a \vee b) = -a \wedge -b$.

c) $a + b = (a \wedge b) + (a \vee b)$.

d) $|a| = a \vee -a = a^+ + a^-$.

e) $|a + b| \leq |a| + |b|$.

f) [**BKW**], (1.3.2, 1.3.3, p. 22) $a = a^+ - a^-$ *and* $a^+ \wedge a^- = 0$.

g) ([**BKW**], Proposition 1.3.4, p. 22) *For x, y, $z \in A$, the following are equivalent:*

(1) $x = y - z$ *and* $y \wedge z = 0$;

(2) $y = x^+$ *and* $z = x^-$. ■

2. *f*-rings. A lattice-ordered ring is called an ***f*-ring** if it is isomorphic to a subdirect product of linearly ordered rings ([**BKW**], Definition 9.1.1, p. 172). We have

FACT 8.3. *Let A be a ring.*

a) ([**BKW**], Proposition 9.1.10, p. 175) *If A is an f-ring and a, b, $x \in A$, then*

(1) $x \geq 0 \Rightarrow x(a \vee b) = xa \vee xb$ *and* $x(a \wedge b) = xa \wedge xb$.

(2) $|ab| = |a| \cdot |b|$.

(3) $a \wedge b = 0 \Rightarrow ab = 0$. (4) $a^2 \geq 0$.

b) ([**BKW**], Corollary 9.1.14, p. 176) *In a ℓ-ring A, properties* $(a.1)$ *and* $(a.2)$ *are equivalent. Moreover, any unitary ring verifying* $(a.1)$ *is an f-ring.*

c) ([**BKW**], Theorem 9.3.1, pp. 178–179) *If A is a ℓ-ring, the following are equivalent:*

(1) *A is a reduced f-ring;*

(2) *A is a subdirect product of linearly ordered integral domains.*

(3) *For all a, $b \in A$,* $|a| \wedge |b| = 0$ *iff* $ab = 0$. ■

[1] For the definition and basic properties of ℓ-groups see Chapter 1 of [**BKW**]. The laws that follow will be used only for the (commutative) additive group of a ℓ-ring.

8.4. **Unit-reflecting preorders** As noted in the preamble to this Chapter the notion of *unit-reflecting preorder*, defined below, will be quite important in the sequel. Notation is as in 4.12.

LEMMA 8.5. *If $\langle A, T \rangle$ is a p-ring and $Y_T = \mathrm{Sper}(A,T)$, the following are equivalent:*

(1) $A^\times \ \cap \ \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha) \ = \ T^\times$;

(2) *For some non-empty $K \subseteq Y_T$,* $A^\times \ \cap \ \bigcap_{\alpha \in K} \alpha \setminus (-\alpha) \ = \ T^\times$.

(3) *For some non-empty $D \subseteq Y_T^*$,* $A^\times \ \cap \ \bigcap_{\beta \in D} \beta \setminus (-\beta) \ = \ T^\times$.

PROOF. Clearly, (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2). Further, since

$$T^\times \ \subseteq \ A^\times \ \cap \ \bigcap_{\alpha \in Y_T} \alpha \setminus (-\alpha) \ \subseteq \ A^\times \cap \ \bigcap_{\alpha \in K} \alpha \setminus (-\alpha),$$

for $K \subseteq Y_T$, we obtain (2) $\Rightarrow$ (1). It remains to see that (2) $\Rightarrow$ (3). With notation as in 4.13.(b), given K as in (2), set $D = \rho_T[K]$. Then, $\emptyset \neq D \subseteq Y_T^*$. Note that

(I) $$\text{For all } \alpha \in Y_T \text{ and } a \in A^\times, \ \ a \in \alpha \ \Leftrightarrow \ a \in \rho_T(\alpha).$$

Indeed, since $\alpha \subseteq \rho_T(\alpha)$, ($\Rightarrow$) in (I) is clear. For the converse, suppose $a \in A^\times \cap \rho_T(\alpha)$, but $a \notin \alpha$; then, $-a \in \alpha \subseteq \rho_T(\alpha)$, whence $a \in \mathrm{supp}(\rho_T(\alpha))$, an impossibility because $\mathrm{supp}(\rho_T(\alpha))$ is a proper prime ideal. Now, (I) implies

$$\begin{aligned} A^\times \ \cap \ \bigcap_{\alpha \in K} \alpha \setminus (-\alpha) \ &= \ A^\times \ \cap \ \bigcap_{\alpha \in K} \rho_T(\alpha) \setminus (-\rho_T(\alpha)) \\ &= \ A^\times \ \cap \ \bigcap_{\beta \in D} \beta \setminus (-\beta), \end{aligned}$$

from which (2) $\Rightarrow$ (3) follows immediately. ■

DEFINITION 8.6. *A preorder T of a ring A is* **unit-reflecting (u.r.)** *if it satisfies the equivalent conditions in Lemma* 8.5. ■

REMARK 8.7. As proven in Proposition 7.2, if $\langle A, T \rangle$ is a p-ring with bounded inversion, then T is unit-reflecting. Moreover, as noted in 7.3, Corollary 4.1.9 in p. 33 of [**Wa**] guarantees that any preorder on a ring with many units is unit-reflecting. Examples of unit-reflecting preorders that are not of bounded inversion are given in Lemma 8.28. An example of a non-unit reflecting preorder is given in 8.32. ■

2. Units and Orthogonal Decompositions of an f-ring

8.8. DEFINITION AND NOTATION. *a*) The partial order of a f-ring A will be denoted by $T_\sharp^A$. If A is clear from context, we use $T_\sharp$ in place of $T_\sharp^A$. Let

$$Y_\sharp = \mathrm{Sper}(A, T_\sharp) \quad \text{ and } \quad G_\sharp(A) = A^\times / {T_\sharp}^\times = \{u^{T_\sharp} : u \in A^\times\}$$

be the real spectrum of $\langle A, T_\sharp \rangle$ and the π-RSG associated to the p-ring $\langle A, T_\sharp \rangle$, respectively. We write $u^\sharp$ instead of $u^{T_\sharp}$ for the elements of $G_\sharp(A)$. For $a \in A$, with notation as in 4.14.4.(*b*), write

$$H^\sharp(a) \ := \ H_A^{T_\sharp}(a) = \{\alpha \in Y_\sharp : \pi_\alpha(a) >_\alpha 0\} = [\![\overline{a} = 1]\!]_\sharp;$$

similarly for $[\![\overline{a} = -1]\!]_\sharp$. If I is an ideal in A, write $\sqrt[\natural]{I}$ for the $T_\sharp$-radical of I in A (cf. 4.14.3).

b) A $\mathbb{Q}$**-function algebra (**$\mathbb{Q}$**FA)** is an f-ring which is also a $\mathbb{Q}$-algebra.

c) An f-ring A is a **Σ-function ring (ΣFR)** if $T_\sharp = \Sigma A^2$. A is a **Σ-function algebra (ΣFA)** if it is both a $\mathbb{Q}$-algebra and a ΣFR. ■

Remarks 8.9. a) By Fact 4.17.(a), an f-ring A is $T_\sharp$-reduced. Since $\Sigma A^2 \subseteq T_\sharp$, and $\sqrt[\mathrm{re}]{0} \subseteq \sqrt[\natural]{0} = \{0\}$, A is a real ring and ΣA^2 is a *partial order* of A.

b) Every ΣFR with weak bounded inversion is a $\mathbb{Q}$-algebra and so a ΣFA. ■

Lemma 8.10. *Let A be a ring and let $e \neq 0$ be an idempotent in A. If A is an f-ring, a $\mathbb{Q}$FA, a ΣFR or a ΣFA, the same is true of Ae.*

Proof. All conclusions are straightforward once it is shown that the property of being an f-ring is inherited by Ae. Fix $e \in B(A)$. We have A as a subdirect product of linearly ordered domains, $A \subseteq \prod_{i\in I} D_i$. Since the idempotents in any linearly ordered ring are just 0 and 1, it follows that $\pi_i(e) \in \{0, 1\}$, for all $i \in I$, where π_i are the coordinate projections. Let $J = \{i \in I : \pi_i(e) = 1\}$; it is straightforward that Ae is a subdirect product of the D_j, $j \in J$, as needed. ■

Proposition 8.11. *Let A be an f-ring and let $Y_\sharp = \mathrm{Sper}(A, T_\sharp)$.*

a) ([**DeMa**], *Lemma* 2.2.) *For each $\alpha \in Y_\sharp$, $\pi_\alpha : A \longrightarrow A_\alpha$ is a morphism of ℓ-rings. Writing $\leq$ for $\leq_{T_\sharp}$, for all $a, b \in A$, we have:*

$$(1)\ a \leq b \;\Rightarrow\; \begin{cases} (i) & a \in \alpha \;\Rightarrow\; b \in \alpha; \\ (ii) & a \in \alpha \setminus (-\alpha) \;\Rightarrow\; b \in \alpha \setminus (-\alpha); \end{cases}$$

(2) $a \in \alpha \;\Rightarrow\; a^- \in \mathrm{supp}(\alpha)$; (3) $a, b \in \alpha \;\Rightarrow\; a \wedge b \in \alpha$;

(4) $a \vee b \in \alpha \;\Rightarrow\; a \in \alpha$ *or* $b \in \alpha$.

b) If T is a preorder of A containing $T_\sharp$, then for all $a, b \in A$,

$$H^T(a \vee b) = H^T(a) \cup H^T(b) \quad \textit{and} \quad H^T(a \wedge b) = H^T(a) \cap H^T(b).$$

In particular, the family $\{H^T(a) \subseteq \mathrm{Sper}(A, T) : a \in T\}$ is a **basis** *for the (spectral) topology on* $\mathrm{Sper}(A, T)$.

Proof. We comment only on (b). Fix $\alpha \in \mathrm{Sper}(A, T) \subseteq Y_\sharp$; since A_α is linearly ordered by $\leq_\alpha$, note that for $a, b \in A$

(i) $\pi_\alpha(a \vee b) = \pi_\alpha(a) \vee \pi_\alpha(b) >_\alpha 0 \Leftrightarrow \pi_\alpha(a) >_\alpha 0$ or $\pi_\alpha(b) >_\alpha 0$;

(ii) $\pi_\alpha(a \wedge b) = \pi_\alpha(a) \wedge \pi_\alpha(b) >_\alpha 0 \Leftrightarrow \pi_\alpha(a) >_\alpha 0$ and $\pi_\alpha(b) >_\alpha 0$.

Now, (i) yields $H^T(a \vee b) = H^T(a) \cup H^T(b)$, while (ii) entails $H^T(a \wedge b) = H^T(a) \cap H^T(b)$; since $\{H^T(a) : a \in A\}$ is a sub-basis for the spectral topology on $\mathrm{Sper}(A, T)$, these equalities imply that it is closed under finite unions and intersections, and therefore a basis for this topology. ■

8.12. **Remarks and Notation.** a) Let A be an f-ring. Since $1 \in (A^2)^\times$, we know that $|1| = 1$. Moreover, if $u \in A^\times$, then Fact 8.3.(a.2) guarantees that $|u| \in A^\times$, and $1/|u| = |1/u|$.

b) The operations of a Boolean algebra, B, are denoted, as usual, by: $\wedge$ (meet), $\vee$ (join), $\triangle$ (symmetric difference) and $-$ (complement); $\perp$ (bottom) and $\top$ (top) are, respectively, the smallest and largest elements of B; $\leq$ denotes the partial order of B.

Recall ([**DM2**], Chapter 5) that any Boolean algebra, B, has a natural structure of *reduced special group* under the group operation $\triangle$, where $1 = \perp$, $-1 = \top$ and $a \in D_B(1, b) \Leftrightarrow a \leq b$.

In particular, if A is a ring and $B(A)$ is the Boolean algebra of its idempotents (cf. 4.1), $\langle B(A), \triangle, 0, 1 \rangle$ is a RSG. If Z is a topological space, $B(Z)$ is the Boolean algebra of its clopens, and $\langle B(Z), \triangle, \emptyset, Z \rangle$ is a RSG. We shall use these facts without further comment. ■

Our next result will be crucial in all that follows.

THEOREM 8.13. *Let A be an f-ring. Each $u \in A^{\times}$ yields an orthogonal decomposition of A into idempotents, $\{\mathfrak{e}^+(u), \mathfrak{e}^-(u)\}$, given by*

$$\mathfrak{e}^+(u) \;=\; \frac{u^+}{|u|} \quad \textit{and} \quad \mathfrak{e}^-(u) \;=\; \frac{u^-}{|u|},$$

with the following properties:

a) (1) $H^{\sharp}(\mathfrak{e}^-(u)) \;=\; H^{\sharp}(u^-) \;=\; [\![\overline{u} = -1]\!]_{\sharp}$ *and*
$H^{\sharp}(\mathfrak{e}^+(u)) \;=\; H^{\sharp}(u^+) \;=\; [\![\overline{u} = 1]\!]_{\sharp}$.

(2) $\mathfrak{e}^-(1) = 0$ *and* $\mathfrak{e}^-(-1) = 1$.

(3) $u \;=\; |u|(\mathfrak{e}^+(u) \;-\; \mathfrak{e}^-(u))$.

(4) $\mathfrak{e}^+(-u) \;=\; \mathfrak{e}^-(u)$ *and* $\mathfrak{e}^-(-u) \;=\; \mathfrak{e}^+(u)$.

(5) $u\mathfrak{e}^+(u) \;=\; |u|\mathfrak{e}^+(u)$ *and* $u\mathfrak{e}^-(u) \;=\; -|u|\mathfrak{e}^-(u)$.

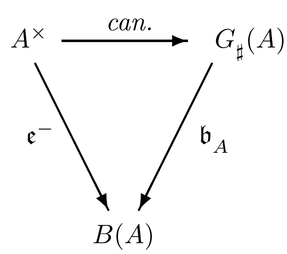
(6) (Uniqueness) *If* $u \;=\; x(e_1 \;-\; e_2)$, *where* $x \geq 0$ *and* e_1, e_2 *are orthogonal idempotents in A, then* $x \;=\; |u|$, $e_1 \;=\; \mathfrak{e}^+(u)$ *and* $e_2 \;=\; \mathfrak{e}^-(u)$.

b) For all u, $v \in A^{\times}$, $\mathfrak{e}^-(uv) \;=\; \mathfrak{e}^-(u) \triangle \mathfrak{e}^-(v)$, *in the Boolean algebra $B(A)$.*

c) The map $\mathfrak{e}^- : A^{\times} \longrightarrow B(A)$, *given by* $u \longmapsto \mathfrak{e}^-(u)$, *is a surjective group morphism taking -1 to 1 ($= \top_{B(A)}$), whose kernel is ${T_{\sharp}}^{\times}$. Moreover, for u, $v \in A^{\times}$,*

(♮) $$u \in D^{v}_{T_{\sharp}}(1, v) \quad \Leftrightarrow \quad \mathfrak{e}^-(u) \leq \mathfrak{e}^-(v).$$

d) Let $G_{\sharp}(A) \;=\; A^{\times}/{T_{\sharp}}^{\times}$ *be the π-RSG associated to $\langle A, T_{\sharp} \rangle$. There is a unique* isomorphism *of π-RSGs,* $\mathfrak{b}_A : G_{\sharp}(A) \longrightarrow B(A)$, *making the following diagram commutative:*



In particular, $G_{\sharp}(A)$ is a reduced special group.

e) Let $n \geq 1$ be an integer and let $x_1, \ldots, x_n \in A^{\times}$. Then, there is an orthogonal decomposition of A, $\{e_1, \ldots, e_m\}$, such that for each $1 \leq k \leq n$ and each $1 \leq j \leq m$:

(1) $e_j \leq \mathfrak{e}^+(x_k)$ *or* $e_j \leq \mathfrak{e}^-(x_k)$;

(2) $$\frac{x_k}{|x_k|} \cdot e_j \;=\; \begin{cases} e_j & \textit{if } e_j \leq \mathfrak{e}^+(x_k); \\ -e_j & \textit{if } e_j \leq \mathfrak{e}^-(x_k). \end{cases}$$

The set $\{e_1, \ldots, e_m\}$ is the **orthogonal decomposition** *of A* **subordinate to** *$x_1, \ldots, x_n$.*

PROOF. For $u \in A^\times$, we have $|u| = u^+ + u^-$, with $u^+u^- = 0$ (8.2.(d) and (f)). Hence,

$$\frac{u^+}{|u|} \cdot \frac{u^-}{|u|} = \frac{u^+u^-}{|u|} = 0 \quad \text{and} \quad \frac{u^+}{|u|} + \frac{u^-}{|u|} = \frac{|u|}{|u|} = 1,$$

and so $\mathfrak{e}^+(u)$ and $\mathfrak{e}^-(u)$ furnish an orthogonal decomposition of A into idempotents.

a) Since $|u|$ is a positive unit in A, (a.1) is clear; (a.2) follows from $1^- = -1 \vee 0 = 0$ and $(-1)^- = 1 \vee 0 = 1$, while (a.3) is an immediate consequence of $u = u^+ - u^-$. For (a.4), we have $-u = u^- - u^+$, with $u^- \wedge u^+ = 0$ and so 8.2.(g) entails $(-u)^+ = u^-$ and $(-u)^- = u^+$. Since $|-1| = 1$, 8.3.(a.2) yields $|-u| = |u|$ and we obtain

$$\mathfrak{e}^+(-u) = \frac{(-u)^+}{|-u|} = \frac{u^-}{|u|} = \mathfrak{e}^-(u);$$

similarly, we prove $\mathfrak{e}^-(-u) = \mathfrak{e}^+(u)$. Item (a.5) follows immediately from (a.3) and the orthogonality of $\mathfrak{e}^+(u)$, $\mathfrak{e}^-(u)$.

(a.6) We have $u = x(e_1 - e_2)$, with $e_1e_2 = 0$ and $xe_1, xe_2 \geq 0$. Hence, $(xe_1)(xe_2) = 0$ and 8.3.(c) and 8.2.(g) imply

$$(i)\quad u^+ = xe_1 \quad \text{and} \quad u^- = xe_2; \qquad (ii)\quad |u| = x(e_1 + e_2) \in A^\times.$$

From (ii) we get $e_1 + e_2 \in A^\times$ and $x \in A^\times$. Let $v \in A$ satisfy $v(e_1 + e_2) = 1$. Then, since $e_1e_2 = 0$, scaling successively by e_1 and e_2, we get $ve_1 = e_1$ and $ve_2 = e_2$, whence $e_1 + e_2 = 1$. Thus, $\{e_1, e_2\}$ is an orthogonal decomposition of A, and (ii) yields $|u| = x$. From (i) we obtain $\mathfrak{e}^+(u)e_2 = \dfrac{u^+e_2}{|u|} = \dfrac{xe_1e_2}{|u|} = 0$; similarly, $\mathfrak{e}^-(u)e_1 = 0$. Hence, $\mathfrak{e}^+(u)e_1 = \mathfrak{e}^+(u)$ and $\mathfrak{e}^-(u)e_2 = \mathfrak{e}^-(u)$, and so

$$e_1 = \mathfrak{e}^+(u)e_1 + \mathfrak{e}^-(u)e_1 = \mathfrak{e}^+(u)e_1 = \mathfrak{e}^+(u).$$

Analogously, $\mathfrak{e}^-(u) = e_2$, as needed.

b) It is known that, for u, v in a reduced f-ring A we have

(I) $$(uv)^+ = u^+v^+ + u^-v^- \quad \text{and} \quad (uv)^- = u^+v^- + v^+u^-.$$

Since $|uv| = |u|\ |v|$ (8.3.(a.2)) and $e \bigtriangleup f = e + f - 2ef$ (4.1.a.(iii)), the second equation in (I) entails,

$$\begin{aligned}\mathfrak{e}^-(uv) &= \frac{u^+v^- + v^+u^-}{|uv|} = \mathfrak{e}^+(u)\mathfrak{e}^-(v) + \mathfrak{e}^+(v)\mathfrak{e}^-(u)\\ &= (1 - \mathfrak{e}^-(u))\mathfrak{e}^-(v) + (1 - \mathfrak{e}^-(v))\mathfrak{e}^-(u)\\ &= \mathfrak{e}^-(u) + \mathfrak{e}^-(v) - 2\mathfrak{e}^-(u)\mathfrak{e}^-(v) = \mathfrak{e}^-(u) \bigtriangleup \mathfrak{e}^-(v).\end{aligned}$$

c) In view of (a) and (b), we must prove $\mathfrak{e}^-$ is onto, has kernel ${T_\sharp}^\times$, as well as the equivalence ($\natural$). Given $e \in B(A)$, set $u = (1 - e) - e$; by 4.3.(a) we have $u \in A^\times$. Since e, $(1 - e) \geq 0$ and $e(1 - e) = 0$, 8.3.(c) entails $e \wedge (1 - e) = 0$ and 8.2.(g) guarantees that $u^+ = 1 - e$ and $u^- = e$. Hence, $|u| = e \vee (1 - e) = e + (1 - e) = 1$ and $\mathfrak{e}^-(u) = e$, showing that $\mathfrak{e}^-$ is onto. Next, for $x \in A^\times$ we have

(II) $$\mathfrak{e}^-(x) = \frac{x^-}{|x|} = 0 \Leftrightarrow x^- = 0 \Leftrightarrow x = x^+ = |x| > 0.$$

Since the positive cone of the partial order in A is $T_\sharp$, (II) entails $x \in \ker \mathfrak{e}^-$ iff $x \in {T_\sharp}^\times$, as desired. It remains to prove ($\natural$).

<u>($\Rightarrow$) :</u> Suppose $u = s_1 + s_2v$, with $s_1, s_2 \in T_\sharp$. Applying (a.3) to u and v, we have

$$|u|(\mathfrak{e}^+(u) - \mathfrak{e}^-(u)) \; = \; u \; = \; s_1 + s_2 v \; = \; s_1 + s_2 \cdot |v|(\mathfrak{e}^+(v) - \mathfrak{e}^-(v));$$

multiplying through by $\mathfrak{e}^+(v)$ yields

$$\text{(III)} \qquad |u|\mathfrak{e}^+(u)\mathfrak{e}^+(v) - |u|\mathfrak{e}^-(u)\mathfrak{e}^+(v) \; = \; (s_1 + |v|s_2)\mathfrak{e}^+(v) \; \geq \; 0.$$

Now observe that $|u|\mathfrak{e}^+(u)\mathfrak{e}^+(v)$, $|u|\mathfrak{e}^-(u)\mathfrak{e}^+(v) \geq 0$ and

$$(|u|\mathfrak{e}^+(u)\mathfrak{e}^+(v))\,(|u|\mathfrak{e}^-(u)\mathfrak{e}^+(v)) \; = \; |u|\mathfrak{e}^+(v)(\mathfrak{e}^+(u)\mathfrak{e}^-(u)) \; = \; 0,$$

and so (III) and 8.2.(g) entail $|u|\mathfrak{e}^-(u)\mathfrak{e}^+(v) = [(s_1 + |v|s_2)\mathfrak{e}^+(v)]^- = 0$, wherefrom, since $|u| \in A^\times$, we conclude $\mathfrak{e}^-(u)\,\mathfrak{e}^+(v) = 0$, or equivalently, $\mathfrak{e}^-(u) \leq \mathfrak{e}^-(v)$.

<u>$(\Leftarrow)$:</u> Assume that $\mathfrak{e}^-(u) \leq \mathfrak{e}^-(v)$, i.e., $\mathfrak{e}^-(u)\mathfrak{e}^+(v) = 0$. We claim that

$$u \; = \; |u|\mathfrak{e}^+(u) + \left(\frac{|u|}{|v|}\mathfrak{e}^-(u)\right)v,$$

and so $u \in D^v_{T_\sharp}(1, v)$, because $|u|\mathfrak{e}^+(u)$ and $\left(\frac{|u|}{|v|}\mathfrak{e}^-(u)\right)$ being positive in A, are in $T_\sharp$. Indeed, recalling (a.3), since $\mathfrak{e}^-(u)\mathfrak{e}^-(v) = \mathfrak{e}^-(u)$ and $\mathfrak{e}^-(u)\mathfrak{e}^+(v) = 0$, we obtain

$$\begin{aligned}
|u|\mathfrak{e}^+(u) + \left(\frac{|u|}{|v|}\mathfrak{e}^-(u)\right)v \; &= \; |u|\mathfrak{e}^+(u) \; + \; \frac{|u|}{|v|}\mathfrak{e}^-(u)|v|(\mathfrak{e}^+(v) - \mathfrak{e}^-(v)) \\
&= \; |u|\mathfrak{e}^+(u) \; + \; |u|\mathfrak{e}^-(u)(\mathfrak{e}^+(v) - \mathfrak{e}^-(v)) \\
&= \; |u|\mathfrak{e}^+(u) - |u|\mathfrak{e}^-(u)\mathfrak{e}^-(v) \\
&= \; |u|(\mathfrak{e}^+(u) - \mathfrak{e}^-(u)) \; = \; u,
\end{aligned}$$

as claimed.

d) The Fundamental Theorem of group morphisms and (c) yield a unique group isomorphism, $\mathfrak{b}_A$, making the displayed diagram commutative, while the second equation in (a.2) guarantees that $\mathfrak{b}_A(-1) = 1$. Recall (see 2.7.(a.1)) that if $G_\sharp(A) = A^\times/{T_\sharp}^\times = \{u^\sharp : u \in A^\times\}$, then

$$u^\sharp \in D_{T_\sharp}(1, v^\sharp) \quad \Leftrightarrow \quad u \in D^v_{T_\sharp}(1, v).$$

Since $\mathfrak{b}_A(u^\sharp) = \mathfrak{e}^-(u)$, Definition 1.14, Remark 1.15 and ($\natural$) in (c) imply $\mathfrak{b}_A$ is an isomorphism of π-RSGs. Since $B(A)$ is a reduced special group, so is $G_\sharp(A)$.

e) Let $\underline{\boldsymbol{n}} = \{1, \ldots, n\}$ and let $\mathcal{P}(n)$ be the power set of $\underline{\boldsymbol{n}}$. For $P \in \mathcal{P}(n)$, set (with the empty product equal to 1)

$$e_P \; = \; \prod_{i\in P} \mathfrak{e}^+(x_i) \cdot \prod_{l\notin P} \mathfrak{e}^-(x_l).$$

Clearly, for all P, $P' \in \mathcal{P}(n)$ and $1 \leq k \leq n$ (recalling that $f \leq f'$ in $B(A) \Leftrightarrow ff' = f$),

$$(+) \quad P \neq P' \;\Rightarrow\; e_P e_{P'} = 0; \qquad\qquad (++) \quad \begin{cases} e_P \leq \mathfrak{e}^+(x_k) & \Leftrightarrow \quad k \in P; \\ e_P \leq \mathfrak{e}^-(x_k) & \Leftrightarrow \quad k \notin P, \end{cases}$$

and these alternatives are mutually exclusive.

Since $1 = \mathfrak{e}^+(x_k) + \mathfrak{e}^-(x_k)$ for $1 \leq k \leq n$, we have

$$1 \; = \; \prod_{k=1}^n (\mathfrak{e}^+(x_k) + \mathfrak{e}^-(x_k)) \; = \; \sum_{P\in\mathcal{P}(n)} e_P,$$

and it follows from (+) above that the collection of e_P *distinct from* 0 constitute an orthogonal decomposition of A into idempotents, to be written $\{e_1, \ldots, e_m\}$. Furthermore, (++) above immediately yields item (e.1). For (e.2), we have, for each $1 \leq k \leq n$ and each $1 \leq j \leq m$, recalling (a.3), (++) above, as well as that $\mathfrak{e}^+(x_k)\mathfrak{e}^-(x_k) = 0$,

$$\frac{x_k}{|x_k|} \cdot e_j \;=\; (\mathfrak{e}^+(x_k) - \mathfrak{e}^-(x_k))\, e_j \;=\; \begin{cases} \mathfrak{e}^+(x_k)e_j = e_j & \text{if } e_j \leq \mathfrak{e}^+(x_k); \\ -\mathfrak{e}^-(x_k)e_j = -e_j & \text{if } e_j \leq \mathfrak{e}^-(x_k), \end{cases}$$

as needed to complete the proof. ■

REMARK 8.14. Let $x_1, \ldots, x_n \in A^\times$. It follows straightforwardly from items (a.4), (b) and the proof of item (e) of Theorem 8.13 that if $t_1, \ldots, t_n \in A^\times$, then the orthogonal decompositions of A into idempotents associated to $\{|t_1|x_1, \ldots, |t_n|x_n\}$ and $\{-|t_1|x_1, \ldots, -|t_n|x_n\}$ constructed therein are the same. ■

Item (a) of the next Proposition is a generalization of a well-known result for rings of continuous real-valued functions:

PROPOSITION 8.15. *Let A be an f-ring and let T be a preorder of A containing $T_\sharp$.*

a) ([**SM**], Prop. 1.11) *If T is a partial order of A, then $T = T_\sharp$. In particular:*

(1) *If A is a ΣFR (cf. 8.8.(c)), ΣA^2 is the only partial order of A.*

(2) *An integral domain is a ΣFR $\Leftrightarrow$ sums of squares is its only linear order.*

b) If $\langle A, T \rangle$ has bounded inversion [2]*, then $T^\times = T_\sharp{}^\times$. In particular, if A is a ΣFR, then $T^\times = (\Sigma A^2)^\times$.*

PROOF. We only prove (b). It suffices to check that $T^\times \subseteq T_\sharp{}^\times$. For $t \in T^\times$, we claim that $\mathfrak{e}^-(t) = 0$. Indeed, since $T \subseteq \alpha$, for all $\alpha \in Y_T$, 8.13.(a.1) yields $H^T(\mathfrak{e}^-(t)) = H^T(-t) = \emptyset$. But then 4.18.(c) entails $\mathfrak{e}^-(t) = 0$, whence, $\mathfrak{e}^+(t) = 1$, i.e., by 8.13.(a.3), $t = |t| \geq 0$ and $t \in T_\sharp{}^\times$, as needed. ■

REMARK 8.16. By 8.15.(b), if $\langle A, T \rangle$ is a BIR, where A is a f-ring and $T_\sharp \subseteq T$, then $G_\sharp(A) = A/T_\sharp{}^\times = G_T(A) = A^\times/T^\times$. Since in BIRs value representation is transversal (7.6.(d)), by Theorem 3.9.(b), to prove that a BIR $\langle A, T \rangle$ is T-faithfully quadratic, with $T_\sharp \subseteq T$ and A an f-ring, it suffices to show A to be $T_\sharp$-faithfully quadratic. Moreover, the RSG associated to such a $\langle A, T \rangle$ is just $G_\sharp(A)$. This of course applies, *ipsis litteris*, to ΣFRs. In this respect, we also call the reader's attention to item (b) of our next result. ■

Complementing items (b) and (c) in Lemma 4.18, we have

PROPOSITION 8.17. *Let A be an f-ring and $T_\sharp \subseteq T$ be a preorder of A. Let $Y_T = \mathrm{Sper}(A, T)$ and let $B(Y_\sharp)$, $B(Y_T)$ be the Boolean algebras of clopens in $Y_\sharp$ and Y_T, respectively.*

a) If $C \in B(Y_T)$, there is $u \geq 0$ in A such that $H^T(u) = C$ and $[\![\overline{u} = 0]\!]_T = Y_T \setminus C$.

b) If $\langle A, T \rangle$ has bounded inversion, the map $e \in B(A) \longmapsto H^T(e) \in B(Y_T)$ is a Boolean algebra isomorphism. In particular, all three Boolean algebras, $B(A)$, $B(Y_\sharp)$ and $B(Y_T)$, are isomorphic.

[2] And hence, so does $\langle A, T_\sharp \rangle$.

PROOF. a) Fix $C \in B(Y_T)$; since C is (quasi-)compact in Y_T and $\{H^T(a) : a \in A\}$ is a basis of opens for Y_T (closed under finite unions and intersections; cf. 8.11.(b)), we have $C = H^T(b)$, for a suitable $b \in A$. Hence, $Y_T \setminus C = \{\alpha \in Y_T : \pi_\alpha(b) \leq_\alpha 0\}$. Set $u = b \vee 0 = b^+ \geq 0$; if $\alpha \in Y_T$, since π_α is a morphism of lattice-ordered rings (8.11.(a)), we get

$$\pi_\alpha(u) \;=\; \pi_\alpha(b \vee 0) \;=\; \begin{cases} \pi_\alpha(b) \;>_\alpha\; 0 & \text{if } \alpha \in C; \\ 0 & \text{if } \alpha \in Y_T \setminus C, \end{cases}$$

whence $C = H^T(u)$, while $[\![\overline{u} = 0]\!]_T = Y_T \setminus C$, as needed.

b) By Lemma 4.18.(c), it suffices to show that the map in the statement is surjective. Fix $C \in B(Y_T)$; by (a), there are $u, v \geq 0$ in A such that

$$\text{(I)} \qquad \begin{cases} H^T(u) \;=\; C \;=\; [\![\overline{v} = 0]\!]_T \quad \text{and} \\ H^T(v) \;=\; Y_T \setminus C \;=\; [\![\overline{u} = 0]\!]_T. \end{cases}$$

Let $w = u - v$; the equations in (I) yield, for $\alpha \in Y_T$,

$$\pi_\alpha(w) \;=\; \pi_\alpha(u - v) \;=\; \begin{cases} \pi_\alpha(u) >_\alpha 0 & \text{if } \alpha \in C; \\ -\pi_\alpha(v) <_\alpha 0 & \text{if } \alpha \in Y_T \setminus C. \end{cases}$$

Hence, $[\![\overline{w} = 0]\!]_T = \emptyset$ and 7.6.(a) entails $w \in A^\times$. Thus,

$$w \;=\; |w|(\mathfrak{e}^+(w) - \mathfrak{e}^-(w)) \;=\; u - v,$$

and so $|w|\mathfrak{e}^+(w) - u \;=\; |w|\mathfrak{e}^-(w) - v$. Multiplying both sides of this equality first by $\mathfrak{e}^+(w)$, and then by $\mathfrak{e}^-(w)$ furnishes, after transposing,

$$u\mathfrak{e}^+(w) = |w|\mathfrak{e}^+(w) + v\mathfrak{e}^+(w) \quad \text{and} \quad v\mathfrak{e}^-(w) = |w|\mathfrak{e}^-(w) + u\mathfrak{e}^-(w).$$

Since u, v, $|w|$, $\mathfrak{e}^+(w)$, $\mathfrak{e}^-(w) \geq 0$, the preceding equalities imply

$$\text{(II)} \qquad u\mathfrak{e}^+(w) \geq v\mathfrak{e}^+(w) \quad \text{and} \quad v\mathfrak{e}^-(w) \geq u\mathfrak{e}^-(w).$$

If $\alpha \in H^T(\mathfrak{e}^+(w))$, i.e., $\pi_\alpha(\mathfrak{e}^+(w)) = 1$, the first inequality in (II) yields $\pi_\alpha(u) \geq_\alpha \pi_\alpha(v)$, which, in view of (I) and $w \in A^\times$, is possible only if $\alpha \in C$. Hence, $H^T(\mathfrak{e}^+(w)) \subseteq C$. A similar argument, based on the second inequality in (II), shows $H^T(\mathfrak{e}^-(w)) \subseteq Y_T \setminus C$. Since $\{C, Y_T \setminus C\}$ and $\{H^T(\mathfrak{e}^+(w)), H^T(\mathfrak{e}^-(w)\}$ are both partitions of Y_T into clopens, the preceding inclusions imply $C = H^T(\mathfrak{e}^+(w))$ and the map in the statement is surjective, and thus a Boolean algebra isomorphism. The last assertion follows from what has been proven and Remark 8.16. ■

3. Quadratic Faithfulness of $\boldsymbol{f}$-rings

In this section we prove that (reduced) f-rings are $T_\sharp$-faithfully quadratic. The nature of the associated RSGs (Boolean algebras) has already been indicated in item (d) of Theorem 8.13. The fundamental result is:

THEOREM 8.18. *Let A be an f-ring, let $n \geq 2$ be an integer and let a, $b_1, \ldots, b_n \in A^\times$. If*

$$\mathfrak{e}^+(a)\mathfrak{e}^-(b_1)\cdots\mathfrak{e}^-(b_n) \;=\; 0 \;=\; \mathfrak{e}^-(a)\mathfrak{e}^+(b_1)\cdots\mathfrak{e}^+(b_n),$$

then

a) $a \in D^t_{T_\sharp}(b_1, \ldots, b_n)$.

b) For each $1 \leq k \leq n$, there is $c \in {T_\sharp}^\times$ such that (*i*) $u := a - cb_k \in A^{\times}$;

(*ii*) $\mathfrak{e}^+(u) \cdot \prod_{j \neq k} \mathfrak{e}^-(b_j) \;=\; 0 \;=\; \mathfrak{e}^-(u) \cdot \prod_{j \neq k} \mathfrak{e}^+(b_j)$.

Note. The equalities in the statement can easily be rephrased as follows:

$$\begin{cases} \{\alpha \in Y_\sharp : b_1(\alpha) < 0 \wedge \ldots \wedge b_n(\alpha) < 0\} \;\subseteq\; \{\alpha \in Y_\sharp : a(\alpha) < 0\}, \\ \{\alpha \in Y_\sharp : b_1(\alpha) > 0 \wedge \ldots \wedge b_n(\alpha) > 0\} \;\subseteq\; \{\alpha \in Y_\sharp : a(\alpha) > 0\}. \end{cases}$$

Proof. By 8.13.(e), we may fix an orthogonal decomposition of A into non-zero idempotents, $\{e_1, \ldots, e_m\}$, subordinate to $\{a, b_1, \ldots, b_n\}$. To lighten notation, set $\varphi = \langle b_1, \ldots, b_n \rangle$; recall (1.1.(b)) that if $\ell \geq 1$ is an integer, then $\underline{\boldsymbol{\ell}} = \{1, \ldots, \ell\}$.

a) Note that for each $j \in \underline{\boldsymbol{m}}$, ae_j, b_1e_j, $\ldots$, $b_ne_j \in (Ae_j)^{\times}$; moreover, we contend that in order to establish (a) it suffices to show that, for each $j \in \underline{\boldsymbol{m}}$, in the ring Ae_j we have

$$\text{(I)} \qquad ae_j \in D^t_{T_\sharp e_j}(b_1e_j, \ldots, b_ne_j) \;=\; D^t_{T_\sharp e_j}(\varphi e_j),$$

where $T_\sharp e_j = \{te_j : t \in T_\sharp\}$. Indeed, suppose $ae_j = \sum_{k=1}^n t_{jk}b_ke_j$, with $t_{jk} \in (T_\sharp e_j)^{\times}$. For $k \in \underline{\boldsymbol{n}}$, define $t_k = \sum_{j=1}^m t_{jk}e_j$; by 4.3.(a), $t_k \in {T_\sharp}^{\times}$, and

$$a \;=\; \sum_{j=1}^m ae_j \;=\; \sum_{j=1}^m \sum_{k=1}^n t_{kj}b_ke_j \;=\; \sum_{k=1}^n \Big(\sum_{j=1}^m t_{jk}e_j\Big) b_k$$
$$=\; \sum_{k=1}^n t_kb_k,$$

whence $a \in D^t_{T_\sharp}(b_1, \ldots, b_n)$, as claimed.

In the remainder of the proof of (a) fix $j \in \underline{\boldsymbol{m}}$ and, to simplify notation, write e for e_j. By 8.13.(e.2), for each $k \in \underline{\boldsymbol{n}}$ we have

$$\text{(II)} \qquad b_ke \;=\; \begin{cases} |b_k|e & \text{if } e \leq \mathfrak{e}^+(b_k); \\ -|b_k|e & \text{if } e \leq \mathfrak{e}^-(b_k). \end{cases}$$

The alternatives in (II), as well as the disjunction $e \leq \mathfrak{e}^+(a)$ or $e \leq \mathfrak{e}^-(a)$, are mutually exclusive, since $e \neq 0$ and $\mathfrak{e}^+(a)\mathfrak{e}^-(a) = 0 = \mathfrak{e}^+(b_k)\mathfrak{e}^-(b_k)$. Set

$$\mathfrak{p}(\varphi) \;=\; \{k \in \underline{\boldsymbol{n}} : b_ke = |b_k|e\} \quad \text{and} \quad p(\varphi) = \mathrm{card}(\mathfrak{p}(\varphi)).$$

Then, $\underline{\boldsymbol{n}} \setminus \mathfrak{p}(\varphi) \;=\; \{k \in \underline{\boldsymbol{n}} : b_ke = -|b_k|e\}$. With this notation, we have:

Fact. *In the proof of* (I) *we may assume* $p(\varphi) \geq n - p(\varphi)$.

Proof. Suppose $n - p(\varphi) > p(\varphi)$ and consider the family of units $-a, -b_1, \ldots, -b_n$. Then:

- By Remark 8.14, the decompositions associated to $\{a,\ b_1, \ldots, b_n\}$ and $\{-a, -b_1, \ldots, -b_n\}$ are the same;
- Item (a.4) of 8.13 yields, by assumption
$$\mathfrak{e}^+(-a) \cdot \prod_{k=1}^n \mathfrak{e}^-(-b_k) \;=\; \mathfrak{e}^-(a) \cdot \prod_{k=1}^n \mathfrak{e}^+(b_k) \;=\; 0;$$
- $\mathfrak{p}(-\varphi) = \underline{\boldsymbol{n}} \setminus \mathfrak{p}(\varphi)$ and so $p(-\varphi) = n - p(\varphi) > p(\varphi) = n - p(-\varphi)$.

Hence, $-ae \in D^t_{T_\sharp e}(-\varphi e)$, clearly equivalent to $ae \in D^t_{T_\sharp e}(\varphi e)$. □

In view of the Fact, henceforth we assume $p(\varphi) \geq n - p(\varphi)$. Moreover, since φ will remain fixed, write $\mathfrak{p}$ and p for $\mathfrak{p}(\varphi)$ and $p(\varphi)$, respectively. Recall our standing hypothesis $2 \in A^{\times}$.

Case 1: $e \leq \mathfrak{e}^+(a)$. Note that $\mathfrak{p} \neq \emptyset$; otherwise (II) yields

$$e \;=\; e \cdot \mathfrak{e}^+(a) \cdot \textstyle\prod_{k=1}^{n} \mathfrak{e}^-(b_k) \neq 0,$$

contrary to assumption; so $p \geq 1$. Moreover, by 8.13.(e.2), $ae = |a|e$.

Case 1.1: $p > n - p$. We partition $\mathfrak{p}$ into disjoint sets, α, β, with $\mathrm{card}(\beta) = n - p$; write ℓ for $\mathrm{card}(\alpha) = 2p - n > 0$. By Fact 4.9, for $1 \leq i \leq \ell$, there are $s_i = 1/2^{k_i}$ in $(\Sigma A^2)^\times \subseteq T_\sharp{}^\times$ such that $1 = \Sigma_{i=1}^{\ell}\, s_i$. Now define, for $k \in \underline{\boldsymbol{n}}$,

$$d_k \;=\; \begin{cases} (|a|e)/|b_k| & \text{if } k \in \beta \cup (\underline{\boldsymbol{n}} \setminus \mathfrak{p}); \\ (|a|s_k e)/|b_k| & \text{if } k \in \alpha. \end{cases}$$

Note that each d_k is a positive unit in Ae and so $d_k \in (T_\sharp e)^\times$. Moreover, since $\mathrm{card}(\beta) = n - p = \mathrm{card}(\underline{\boldsymbol{n}} \setminus \mathfrak{p})$, with $b_k e = |b_k|e$ if $k \in \beta \subseteq \mathfrak{p}$ and $b_k e = -|b_k|e$ if $k \in \underline{\boldsymbol{n}} \setminus \mathfrak{p}$, it is clear that $\Sigma_{k\in\beta}\, d_k b_k = (n-p)|a|e$ and $\Sigma_{k\notin\mathfrak{p}}\, d_k b_k = -\,(n-p)|a|e$. Hence,

$$\begin{aligned} \textstyle\sum_{k=1}^{n} d_k b_k e \;&=\; \textstyle\sum_{k\in\alpha} \dfrac{|a|s_k}{|b_k|}\, b_k e \;+\; \textstyle\sum_{k\in\beta} d_k b_k \;+\; \textstyle\sum_{k\notin\mathfrak{p}} d_k b_k \\ &=\; \textstyle\sum_{k\in\alpha} \dfrac{|a|s_k}{|b_k|} \cdot |b_k|e \;=\; \textstyle\sum_{k\in\alpha} |a|e s_k \;=\; |a|e \textstyle\sum_{k\in\alpha} s_k \\ &=\; |a|e \;=\; ae \end{aligned}$$

and so $ae \in D^t_{T_\sharp e}(\varphi e)$, as needed.

Case 1.2: $p = n - p$. Set $k_0 = \min \mathfrak{p}$ and define, for $k \in \underline{\boldsymbol{n}}$,

$$d_k \;=\; \begin{cases} (2|a|e)/|b_{k_0}| & \text{if } k = k_0; \\ (|a|e)/|b_k| & \text{if } k \neq k_0. \end{cases}$$

As in Case 1.1, each d_k is a unit in $T_\sharp e$. Since $\mathrm{card}(\mathfrak{p} \setminus \{k_0\}) = p - 1$ and $\mathrm{card}(\underline{\boldsymbol{n}} \setminus \mathfrak{p}) = p$, with $b_k e = |b_k|e$ if $k \in (\mathfrak{p} \setminus \{k_0\})$, and $b_k e = -|b_k|e$ if $k \notin \mathfrak{p}$, it follows that $\Sigma_{k\in\mathfrak{p},k\neq k_0}\, d_k b_k = (p-1)|a|e$, while $\Sigma_{k\notin\mathfrak{p}}\, d_k e_k = -\,p|a|e$. Hence,

$$\begin{aligned} \textstyle\sum_{k=1}^{n} d_k b_k e \;&=\; \dfrac{2|a|}{|b_{k_0}|}\, b_{k_0} e \;+\; \textstyle\sum_{k\in\mathfrak{p},k\neq k_0} d_k b_k \;+\; \textstyle\sum_{k\notin\mathfrak{p}} d_k b_k \\ &=\; 2|a|e - |a|e \;=\; |a|e \;=\; ae, \end{aligned}$$

as needed to establish $ae \in D^t_{T_\sharp e}(\varphi e)$.

Case 2: $e \leq \mathfrak{e}^-(a)$. Since $\mathfrak{e}^-(a) \cdot \prod_{k=1} \mathfrak{e}^+(b_k) = 0$ and $e \neq 0$, as in Case 1 we must have $\mathfrak{p} \neq \underline{\boldsymbol{n}}$, i.e., $n - p \geq 1$. Moreover, by item (e.2) in 8.13, $ae = -|a|e$. As in Case 1, we consider two situations, giving explicit expressions for the corresponding coefficients d_k, $k \in \underline{\boldsymbol{n}}$, and leaving details to the reader.

Case 2.1: $p > n - p$. Using Fact 4.9 we choose two representations of 1 as sums of elements of the form $1/2^k$, of length p and $n - p$, respectively; say $\{s_1, \ldots, s_p\}$ and $\{s'_1, \ldots, s'_{n-p}\}$. We can index these sets by $\mathfrak{p}$ and $\underline{\boldsymbol{n}} \setminus \mathfrak{p}$, respectively, so that $\sum_{k\in\mathfrak{p}} s_k = 1 = \sum_{j\notin\mathfrak{p}} s'_j$. Note that s_k, s'_j are positive units of A. For $k \in \underline{\boldsymbol{n}}$ we define:

$$d_k \;=\; \begin{cases} (|a|s_k e)/|b_k| & \text{if } k \in \mathfrak{p}; \\ (2|a|s'_k e)/|b_k| & \text{if } k \in \underline{\boldsymbol{n}} \setminus \mathfrak{p}. \end{cases}$$

Case 2.2: $p = n - p$. Let $k_0 = \min(\underline{\boldsymbol{n}} \setminus \mathfrak{p})$ and, for $k \in \underline{\boldsymbol{n}}$, define

$$d_k = \begin{cases} (2|a|e)/|b_{k_0}| & \text{if } k = k_0; \\ (|a|e)/|b_k| & \text{if } k \neq k_0, \end{cases}$$

completing the proof of (I) and of item (a).

b) Fix $k \in \underline{\boldsymbol{n}}$. Set $\pi = \prod_{j \neq k} \mathfrak{e}^+(b_j)$ and $\nu = \prod_{j \neq k} \mathfrak{e}^-(b_j)$, clearly orthogonal idempotents in A (since $n \geq 2$); we in turn decompose π and ν into orthogonal idempotents, $\pi = \pi_1 + \pi_2 + \pi_3$ and $\nu = \nu_1 + \nu_2 + \nu_3$, where

$$\begin{cases} \pi_1 = \pi\mathfrak{e}^-(a), & \pi_2 = \pi\mathfrak{e}^+(a)\mathfrak{e}^-(b_k), & \pi_3 = \pi\mathfrak{e}^+(a)\mathfrak{e}^+(b_k); \\ \nu_1 = \nu\mathfrak{e}^+(a), & \nu_2 = \nu\mathfrak{e}^-(a)\mathfrak{e}^+(b_k), & \nu_3 = \nu\mathfrak{e}^-(a)\mathfrak{e}^-(b_k). \end{cases}$$

Moreover, set $f = 1 - (\pi + \nu)$, the complement of $\pi \vee \nu = \pi + \nu$ in $B(A)$. We also decompose f into orthogonal idempotents as follows:

$f_1 = \mathfrak{e}^+(a)\mathfrak{e}^+(b_k)f$, $f_2 = \mathfrak{e}^-(a)\mathfrak{e}^-(b_k)f$, $f_3 = \mathfrak{e}^+(a)\mathfrak{e}^-(b_k)f$, and $f_4 = \mathrm{a}\mathfrak{e}^+(b_k)f$.

Note that f_i, $i = 1, \ldots, 4$, is the decomposition of A associated to $\{a, b_k\}$, intersected with f. Therefore,

$$\{f_i : i \in \underline{\mathbf{4}}\} \cup \{\pi_i : i \in \underline{\mathbf{3}}\} \cup \{\nu_i : i \in \underline{\mathbf{3}}\}$$

is an orthogonal decomposition of A into idempotents. Let

$$d_1^\pi = d_1^\nu = \frac{2|a|}{|b_k|}; \qquad x_3 = x_4 = d_2^\pi = d_2^\nu = \frac{|a|}{|b_k|};$$
$$x_1 = x_2 = d_3^\pi = d_3^\nu = \frac{|a|}{2|b_k|}.$$

Clearly, x_i ($i \in \underline{\mathbf{4}}$) and d_i^π, d_i^ν ($i \in \underline{\mathbf{3}}$), are units in $T_\sharp$. Now set

$$c = \textstyle\sum_{i=1}^4 x_i f_i + \sum_{i=1}^3 d_i^\pi \pi_i + \sum_{i=1}^3 d_i^\nu \nu_i.$$

By item 4.3.(a), $c \in T_\sharp{}^\times$. We claim that c verifies the required conditions (b.i) and (b.ii) in the statement of Theorem 8.18.

Proof of ($\boldsymbol{b.i}$): The various cases are subsumed into the following

FACT 8.19. *Let A be a f-ring, $a, b \in A^\times$, $r \in A$ and g be an idempotent of A. Let $u := a - \dfrac{r|a|}{|b|} \cdot b$. Then, under any of the following conditions* (1) *or* (2), *we have $ug \in (Ag)^\times$, where*

(1) $1 - r \in A^\times$ *and either* (i) $g \leq \mathfrak{e}^+(a)\mathfrak{e}^+(b)$ *or* (ii) $g \leq \mathfrak{e}^-(a)\mathfrak{e}^-(b)$.

Further, the sign of ug in Ag is the same as that of $1 - r$ in case ($1.i$) *and the opposite in case* ($1.ii$).

(2) $1 + r \in A^\times$ *and either* (i) $g \leq \mathfrak{e}^-(a)\mathfrak{e}^+(b)$ *or* (ii) $g \leq \mathfrak{e}^+(a)\mathfrak{e}^-(b)$.

Further, the sign of ug in Ag is the same as that of $1 + r$ in case ($2.ii$) *and the opposite in case* ($2.i$).

PROOF. From 8.13.(a.3) and the definition of u we have

$$(*) \qquad ug = ag - r|a|g(\mathfrak{e}^+(b) - \mathfrak{e}^-(b)).$$

Now we analyze (*) in each of the four cases of the statement.

(1.i) In this case we have $g\mathfrak{e}^+(a) = g = g\mathfrak{e}^+(b)$ and (in view of $\mathfrak{e}^+(b)\mathfrak{e}^-(b) = 0$, cf. 8.13), $g\mathfrak{e}^-(b) = 0$. Replacing these equalities in (*), using 8.13.(a.5) and the assumption $1 - r \in A^\times$, gives

$$\begin{aligned} ug &= a\mathfrak{e}^+(a)g - r|a|g\mathfrak{e}^+(b) = |a|\mathfrak{e}^+(a)g - r|a|g = |a|g - r|a|g \\ &= |a|g(1 - r) \in (Ag)^\times. \end{aligned}$$

(1.ii) Now we have $g\mathfrak{e}^-(a) = g = g\mathfrak{e}^-(b)$ and $g\mathfrak{e}^+(b) = 0$. Proceeding as in the previous case, we obtain

$$\begin{aligned} ug &= a\mathfrak{e}^-(a)g + r|a|g\mathfrak{e}^-(b) = a\mathfrak{e}^-(a)g + r|a|g = -|a|g + r|a|g \\ &= -|a|g(1 - r) \in (Ag)^\times. \end{aligned}$$

(2.i) The assumption gives $g\mathfrak{e}^-(a) = g = g\mathfrak{e}^+(b)$ and $g\mathfrak{e}^-(b) = 0$, whence (*) yields:

$$\begin{aligned} ug &= a\mathfrak{e}^-(a)g - r|a|g\mathfrak{e}^+(b) = -|a|\mathfrak{e}^-(a)g - r|a|g \\ &= -|a|g - r|a|g = -(1 + r)|a|g \in (Ag)^\times. \end{aligned}$$

(2.ii) We have $g\mathfrak{e}^+(a) = g = g\mathfrak{e}^-(b)$ and $g\mathfrak{e}^+(b) = 0$, whence

$$\begin{aligned} ug &= a\mathfrak{e}^+(a)g + r|a|g\mathfrak{e}^-(b) = |a|\mathfrak{e}^+(a)g + r|a|g \\ &= |a|g(1 + r) \in (Ag)^\times. \end{aligned}$$

In all four cases the assertion about signs is clear upon inspection. □

Now, to prove **(*b.i*)** it suffices to apply Fact 8.19 with the following choice of parameters:

– a is the homonymous element in the statement;

– For given $k \in \underline{\boldsymbol{n}}$, $b = b_k$, while r is given as follows:

(● 1) $r = \dfrac{1}{2}$ if $g \in \{f_1, f_2, \pi_3, \nu_3\}$; (● 2) $r = 1$ if $g \in \{f_3, f_4, \pi_2, \nu_2\}$;

(● 3) $r = 2$ if $g \in \{\pi_1, \nu_1\}$.

It remains to be checked that the appropriate value $1 - r$ or $1 + r$, applying to each case, is invertible. By direct inspection of the definition of the f_i and π_j, ν_j, we have:

– In case (● 1), we are in cases (1.i) or (1.ii) of 8.19, whence $1 - r = \dfrac{1}{2} \in A^\times$;

– In case (● 2), we are in cases (2.i) or (2.ii) of 8.19, whence $1 + r = 2 \in A^\times$;

– The case (● 3) requires a little extra argument. For example, if $g = \pi_1 = \pi\mathfrak{e}^-(a)$, from the definition of π we have

$$\pi_1\mathfrak{e}^+(b_k) = \mathfrak{e}^-(a) \textstyle\prod_{j=1}^n \mathfrak{e}^+(b_j) = 0$$

by assumption of the Theorem; hence $\pi_1 \le \mathfrak{e}^-(b_k)$ and we are in case (1.ii) of 8.19; thus, $1 - r = 1 - 2 = -1 \in A^\times$. Likewise, if $g = \nu_1 = \nu\mathfrak{e}^+(a)$, then $g \le \mathfrak{e}^+(b_k)$, and 8.19.(1.$i$) yields $1 - r = -1 \in A^\times$.

Note also that the rules of signs proved in Fact 8.19 show that for $i = 1, 2, 3$, $u\pi_i$ is a positive unit in each of the localizations $A\pi_i$ of the orthogonal decomposition of $A\pi$ given by $\pi = \pi_1 + \pi_2 + \pi_3$. Hence, $u\pi$ is positive unit in $A\pi$ (4.3.(a)), that is, $u\pi \in (T_\sharp\pi)^\times$. Since $u\pi\nu = u\pi f = 0$ and $1 = \pi + \nu + f$, we have

$$u\pi = u\pi + u\pi(1 - \pi) \in T_\sharp\pi + T_\sharp(1 - \pi) = T_\sharp,$$

yielding the first relation in

(III) $\quad u\pi \;\geq\; 0 \quad \text{and} \quad u\nu \;\leq\; 0 \quad$ (in A);

similarly, we obtain $u\nu \leq 0$ in A.

Proof of (*b.ii*): We have

$$u\pi \;=\; |u|(\mathfrak{e}^+(u) - \mathfrak{e}^-(u))\pi \;=\; |u|\mathfrak{e}^+(u)\pi - |u|\mathfrak{e}^-(u)\pi.$$

Since $|u|\mathfrak{e}^+(u)\pi$, $|u|\mathfrak{e}^-(u)\pi \geq 0$ and their product is zero, it follows from 8.3.(c) that their meet is also zero, whence 8.2.(g) entails

(IV) $\quad (u\pi)^+ \;=\; |u|\mathfrak{e}^+(u)\pi \quad \text{and} \quad (u\pi)^- \;=\; |u|\mathfrak{e}^-(u)\pi.$

By (III) above, $u\pi \geq 0$ in A and so its negative part must be zero. Since $u \in A^\times$, the second equation in (IV) entails

$$\mathfrak{e}^-(u)\pi \;=\; \mathfrak{e}^-(u) \cdot \textstyle\prod_{j\neq k} \mathfrak{e}^+(b_j) \;=\; 0.$$

A similar argument, using the second inequality in (III), establishes $\mathfrak{e}^+(u)\nu = \mathfrak{e}^+(u) \cdot \prod_{j\neq k} \mathfrak{e}^-(b_j) = 0$, completing the proof of 8.18. ■

Theorem 8.20. *If A is a f-ring, then A is $T_\sharp$-faithfully quadratic. Moreover, the map $\mathfrak{b}_A : G_\sharp(A) \longrightarrow B(A)$, given by $\mathfrak{b}_A(a^\sharp) = \mathfrak{e}^-(a)$ is an isomorphism of reduced special groups.*

Proof. By Theorem 8.13.(d), the map

$$\mathfrak{b}_A : G_\sharp(A) \longrightarrow B(A), \quad \text{given by } \mathfrak{b}_A(a^\sharp) \;=\; \mathfrak{e}^-(a)$$

is an isomorphism of π-SGs. In particular, $G_\sharp(A)$ is a Boolean algebra, and hence a reduced special group.

A satisfies [$T_\sharp$-FQ 1]. By 2.26.(c), it suffices to verify $D^v_{T_\sharp}(\varphi) \subseteq D^t_{T_\sharp}(\varphi)$. If $\varphi = \langle b_1, \ldots, b_n \rangle$ and $a = \sum_{k=1}^n s_k b_k$, with $s_k \in T_\sharp$, $k \in \underline{\boldsymbol{n}}$, then 8.13.(a.3) entails

(I) $\quad |a|(\mathfrak{e}^+(a) - \mathfrak{e}^-(a)) \;=\; \sum_{k=1}^n s_k |b_k|(\mathfrak{e}^+(b_k) - \mathfrak{e}^-(b_k)).$

Multiplying both sides of (I) by $f = \mathfrak{e}^+(a) \cdot \prod_{k=1}^n \mathfrak{e}^-(b_k)$, we arrive at

(II) $\quad |a|f \;=\; \sum_{k=1}^n -s_k|b_k|f.$

Since $|a|f \geq 0$, while $\sum_{k=1}^n -s_k|b_k|f \leq 0$, and $\leq (= T_\sharp)$ is a partial order, we must have $|a|f = 0$. Since, $|a| \in A^\times$, we obtain $f = 0$. Similarly, one verifies that $\mathfrak{e}^-(a) \cdot \prod_{k=1}^n \mathfrak{e}^+(b_k) = 0$. But then item (a) in Theorem 8.18 entails $a \in D^t_{T_\sharp}(\varphi)$, as desired.

A satisfies [$T_\sharp$-FQ 2]. By 2.26.(c), it is enough to check that $D^v_{T_\sharp}(\varphi) \subseteq \mathfrak{D}_{T_\sharp}(\varphi)$. Assume $a \in D^v_{T_\sharp}(\varphi)$; exactly as in the proof of the previous item, we have

$$\mathfrak{e}^+(a) \cdot \textstyle\prod_{k=1}^n \mathfrak{e}^-(b_k) \;=\; 0 \;=\; \mathfrak{e}^-(a) \cdot \prod_{k=1}^n \mathfrak{e}^+(b_k).$$

Then, item (b) of Theorem 8.18 guarantees that for each $k \in \underline{\boldsymbol{n}}$, there is $c \in T_\sharp^{\times}$ such that $u = a - cb_k \in A^\times$ and $\mathfrak{e}^+(u) \cdot \prod_{j\neq k} \mathfrak{e}^-(b_j) = 0 = \mathfrak{e}^-(u) \cdot \prod_{j\neq k} \mathfrak{e}^+(b_j)$. Hence, $a \in D^t_{T_\sharp}(b_k, u)$ (by the first equality), while the last two entail, by 8.18.(a), $u \in D^t_{T_\sharp}(b_1, \ldots, \check{b}_k, \ldots, b_n)$, proving, in fact, more than required.

As a preliminary step to the proof that A verifies axiom [$T_\sharp$-FQ 3], we make the following observations:

Claim 1. *Given a ring A, let $\emptyset \neq \mathcal{K} \subseteq \mathrm{Sper}(A)$. Then, the preorder $T = \bigcap \mathcal{K}$ is unit-reflecting.*

Proof of Claim 1. Immediate from Lemma 8.5.(2): if $t \in T^{\times}$ and $\alpha \in \mathcal{K}$, then $t \in \alpha$. If $t \in -\alpha$, then $t \in \mathrm{supp}(\alpha)$, a proper ideal of A, contradicting that t is invertible. □

Claim 2. *Let A be an f-ring and let $T_\sharp$ be its natural partial order. Then, $T_\sharp$ is unit-reflecting.*

Proof of Claim 2. Let $\{\langle D_i, \beta_i \rangle : i \in I\}$ be a family of linearly ordered domains such that A is a subdirect product of the $\langle D_i, \beta_i \rangle$. For each $i \in I$, let $p_i : A \longrightarrow D_i$ be the restriction of the coordinate projections π_i from $\prod_{j\in I} D_j$ to D_i, which we know to be surjective, and let $\alpha_i = p_i^{-1}[\beta_i]$. Since, p_i is a ring morphism, $\alpha_i \in \mathrm{Sper}(A)$. Now, observe that for all $a \in A$,

$$a \in T_\sharp \quad \text{iff} \quad \forall\, i \in I\ (p_i(a) \in \beta_i),$$

whence $T_\sharp = \bigcap_{i\in I} \alpha_i$, and, by Claim 1, $T_\sharp$ is a unit-reflecting partial order on A. □

In the statement and proof of items (b) and (c) of our next Claim we employ Proposition 2.9 without explicit mention.

Claim 3. *a) Let $\langle D, \beta \rangle$ be a linearly ordered domain and let k be its field of fractions. Set*

$$T_\beta = \left\{\frac{s}{t^2} \in k : s \in \beta \text{ and } t \neq 0 \text{ in } D\right\}.$$

Let $\nu : D \longrightarrow k$ be the canonical injection. Then,

(1) *T_β is the unique total order of k extending β and ν is a p-ring morphism.*

(2) *The π-SG morphism induced by ν, $\nu^\pi : G_\beta(D) \longrightarrow G_{T_\beta}(k)$, is injective.*

b) Let $\{\langle D_i, \beta_i \rangle : i \in I\}$ be a family of linearly ordered domains and (with notation as in (a)), let $\{\langle k_i, T_{\beta_i} \rangle : i \in I\}$ be the associated family of preordered fields. Let $\nu := \prod_{i\in I} \nu_i : \prod_{i\in I} D_i \longrightarrow \prod_{i\in I} k_i$ be the product of the canonical p-ring morphisms $\nu_i : D_i \longrightarrow k_i$. Then, the SG-morphism induced by ν, $\nu^\pi : \prod_{i\in I} G_{\beta_i}(D_i) \longrightarrow \prod_{i\in I} G_{T_{\beta_i}}(k_i)$, is injective.

c) Let A be an f-ring and let $T_\sharp$ be its natural partial order. Let $\{\langle D_i, \beta_i \rangle : i \in I\}$ be a family of linearly ordered domains such that A is a subdirect product of the $\langle D_i, \beta_i \rangle$. With notation as in (b), let $\gamma : A \longrightarrow \prod_{i\in I} k_i$ be the composition of the natural embedding of A into $\prod_{i\in I} D_i$ with the p-ring morphism ν in item (b). Then, the π-SG morphism, $\gamma^\pi : G_\sharp(A) = G_{T_\sharp}(A) \longrightarrow \prod_{i\in I} G_{T_{\beta_i}}(k_i)$, is injective.

Proof of Claim 3. a) (1) is straightforward and well-known. For (2), in view of the definition of ν^π (see Lemma 2.8), to show it to be injective it suffices to check, for $a \in D^{\times}$,

(*) $$\frac{a}{1} \in T_\beta \quad \Rightarrow \quad a \in \beta.$$

The hypothesis in (*) yields $s \in \beta$ and $t \neq 0$ in D such that $\frac{a}{1} = \frac{s}{t^2}$, i.e., $t^2 a = s \in \beta$, because D is a domain. Note that, since $t \neq 0$ and $a \in D^{\times}$, we must have $s \neq 0$ and so, since β is a linear order, $s \in \beta \setminus -\beta$. But then $a \in \beta$; otherwise $a <_\beta 0$ ($a \in D^{\times}$) and $t^2 >_\beta 0$ entail $s \in -\beta$, an impossibility. Hence, we obtain

$a \in \beta \cap D^{\times} = \beta^{\times}$, establishing (*), as needed. Item (b) follows immediately from (a.2), since ν^{π} is a product of injectives.

c) Let $\eta : A \longrightarrow \prod_{i \in I} D_i$ be an embedding of A into the product of the D_i. The desired result follows immediately from (b), as soon as it is shown that $\eta^{\pi} : G_{\sharp}(A) \longrightarrow \prod_{i \in I} G_{\beta_i}(D_i)$ is injective, or equivalently,

(**) $$\forall\ i \in I,\ \ p_i(a) \in \beta_i \quad \Rightarrow \quad a \in T_{\sharp}.$$

Since, by the proof of Claim 2, we have

$$a \in A^{\times} \cap \bigcap_{i \in I} p_i^{-1}[\beta_i] \ = \ A^{\times} \cap \bigcap_{i \in I} \alpha_i \ = \ {T_{\sharp}}^{\times},$$

(**) follows immediately. □

A verifies [$T_{\sharp}$-FQ 3]. We use Claim 3 and notation therein. Since the (infinite) fields k_i are rings with many units, so is the ring $\prod_{i \in I} k_i$ (see Remark 6.2.(a)). By Theorem 6.5, $\prod_{i \in I} k_i$ is, in particular, $\prod_{i \in I} T_{\beta_i}$- faithfully quadratic. The map γ^{π} in Claim 3.(c) is an injective SG-morphism from the Boolean algebra $G_{\sharp}(A)$ (8.13.(d)) into $\prod_{i \in I} G_{T_{\beta_i}}(k_i)$. By [**DM2**], Cor. 5.6, p. 81, γ^{π} is a complete embedding, whence, by Proposition 3.12, $\langle A, T_{\sharp} \rangle$ satisfies [$T_{\sharp}$-FQ 3], ending the proof. ■

As a consequence of Theorem 3.9, we obtain the following generalization of Theorem 8.20:

THEOREM 8.21. *Let A be a f-ring.*

a) If T is a preorder of A containing $T_{\sharp}$, then A is T-faithfully quadratic. Moreover,

b) $G_T(A)$ is isomorphic to $B(A)/\mathcal{I}$, where $\mathcal{I}$ is the ideal of $B(A)$ generated by $Y_T = \mathrm{Sper}(A, T)$ in $Y = \mathrm{Sper}(A)$, i.e.,

$$\mathcal{I} \ = \ \{e \in B(A) : H(e) \cap Y_T = \emptyset\},$$

with $H : B(A) \longrightarrow B(Y)$ given by (cf. Lemma 4.18.(b)):

$$H(e) = \{\alpha \in \mathrm{Sper}(A) : \pi_{\alpha}(e) = 1\}.$$

PROOF. a) By Lemma 2.26.(c) and Theorems 3.9.(b) and 8.20, it suffices to show that if $\varphi = \langle b_1, \ldots, b_n \rangle$ is a form over $A^{\times}$, $D^v_T(\varphi) \subseteq D^t_T(\varphi)$. Assume $a \in A^{\times}$ satisfies

(I) $$a \ = \ \textstyle\sum_{j=1}^{n} s_j b_j, \ \ \text{with } s_j \in T,\ j \in \underline{\boldsymbol{n}}.$$

Let $\{e_1, \ldots, e_m\}$ be the orthogonal decomposition of A subordinate to $\{a, b_1, \ldots, b_n\}$ constructed in the proof of Theorem 8.13.(e). That construction guarantees that both $f =: \mathfrak{e}^+(a)\mathfrak{e}^-(b_1) \cdots \mathfrak{e}^-(b_n)$ and $g =: \mathfrak{e}^-(a)\mathfrak{e}^+(b_1) \cdots \mathfrak{e}^+(b_n)$, if non-zero, are elements of $\{e_1, \ldots, e_m\}$. Set

$$D \ = \ \{e_k \in \{e_1, \ldots, e_m\} : H(e_k) \cap Y_T \neq \emptyset\},$$

and let $e \ = \ \bigvee_{e_k \in D} e_k \ = \ \sum_{e_k \in D} e_k$. We now note

Fact 1. *With notation as above, f, $g \notin D$, whence $fe = ge = 0$.*

Proof of Fact 1. We prove the statement for f; a similar argument treats the case of g. Assume, to get a contradiction, that there is $\alpha \in H(f) \cap Y_T$. Multiplying equation (I) by f yields

(II) $$af \ = \ \textstyle\sum_{j=1}^{n} s_j b_j f,$$

which is impossible, since, by its definition, $af \in \alpha \setminus (-\alpha)$, while the sum in the right-hand side of (II) is in $-\alpha$. As observed above, f is an element of the orthogonal decomposition $\{e_1, \ldots, e_m\}$; then, it is clear that $fe = 0$. □

An important remark for the proof is: [3]

Fact 2. *Let $\langle R, P' \rangle$ be a p-ring, or $P' = R^2$, and let P be a preorder on R containing P'. Let $Y_{P'} = \mathrm{Sper}(A, P')$ and $Y_P = \mathrm{Sper}(A, P)$. If c is an idempotent in R such that $H(c) \cap Y_P = \emptyset$, then $Rc = Pc$.*

Proof of Fact 2. The hypothesis and the definition of $H(c)$ guarantee that for $\alpha \in Y_P$, $\pi_\alpha(c) = 0$. Hence, $c \in \sqrt[P]{0}$; by 4.14.(3), there are $m \geq 1$ and $p \in P$ such that $0 = c^{2m} + p = c + p$. Thus, $-c$ $(= -1$ in the ring $Rc)$ is in P, and so $Pc = Rc$, as desired. □

It follows from Fact 1 that

(III) $$e \cdot \mathfrak{e}^+(a)\mathfrak{e}^-(b_1) \cdots \mathfrak{e}^-(b_n) \;=\; 0 \;=\; e \cdot \mathfrak{e}^-(a)\mathfrak{e}^+(b_1) \cdots \mathfrak{e}^+(b_n).$$

Now observe:

(IV) For all $z \in A^\times$, *in the ring* Ae, $\mathfrak{e}^+(ze) = \mathfrak{e}^+(z)e$ and $\mathfrak{e}^-(ze) = \mathfrak{e}^-(z)e$.

Indeed, since $|e| = e$ and $z = |z|(\mathfrak{e}^+(z) - \mathfrak{e}^-(z))$, we get, recalling 8.3.(a.2), $ze = |ze|(\mathfrak{e}^+(z)e - \mathfrak{e}^-(z)e)$ and the uniqueness in 8.13.(a.6) immediately entails (IV). Because ae, b_1e, ..., b_ne are units in Ae, and since $e = e^{n+1}$, (III) and (IV) imply

$$\mathfrak{e}^+(ae)\mathfrak{e}^-(b_1e) \cdots \mathfrak{e}^-(b_ne) \;=\; 0 \;=\; \mathfrak{e}^-(ae)\mathfrak{e}^+(b_1e) \cdots \mathfrak{e}^+(b_ne),$$

whence Theorem 8.18.(a) guarantees $ae \in D^t_{T_\sharp e}(b_1e, \ldots, b_ne)$. Thus, there are $v_1, \ldots, v_n \in (T_\sharp e)^\times$ such that $ae = \sum_{j=1}^n v_j b_j e$. By Fact 4.9, there are $x_1, \ldots, x_n \in (\Sigma A^2)^\times$ such that $\sum_{j=1}^n x_j = 1$. For $j \in \underline{\boldsymbol{n}}$, define

$$t_j \;=\; v_j + \frac{ax_j}{b_j}(1-e) \;=\; v_j e + \frac{ax_j}{b_j}(1-e).$$

Fact 2 entails $\dfrac{ax_j}{b_j}(1-e) \in (T(1-e))^\times$. Since $v_j \in (T_\sharp e)^\times$, item (a.2) in Lemma 4.3 yields $t_j \in T^\times$ for all $j \in \underline{\boldsymbol{n}}$, and we have

$$\begin{aligned} \textstyle\sum_{j=1}^n t_j b_j \;&=\; \textstyle\sum_{j=1}^n v_j b_j e + \sum_{j=1}^n \dfrac{ax_j}{b_j} b_j (1-e) \\ &=\; ae + \textstyle\sum_{j=1}^n a x_j (1-e) \;=\; ae + a(1-e) \;=\; a, \end{aligned}$$

completing the proof of (a).

b) To determine $G_T(A)$, let $Y = \mathrm{Sper}(A)$; since $H : B(A) \longrightarrow B(Y)$ is a Boolean algebra morphism (cf. 4.18.(b)), it is straightforward to show $\mathcal{I}$ is an ideal in the Boolean algebra $B(A)$. With notation as in 8.13.(d) and its proof, by Theorem 3.9.(a), $\Delta_T = \{t^\sharp : t \in T^\times\}$ is a saturated subgroup of $G_\sharp(A)$ and $G_T(A) = G_\sharp(A)/\Delta_T$. Since $\mathfrak{b}_A : G_\sharp(A) \longrightarrow B(A)$ is an isomorphism of RSGs (cf. Theorem 8.20), to establish the last assertion it is enough to check that $\mathfrak{b}_A(\Delta_T) = \mathcal{I}$, or equivalently

(V) For all $x \in A^\times$, $\mathfrak{b}_A(x^\sharp) = \mathfrak{e}^-(x) \in \mathcal{I} \Leftrightarrow x^\sharp \in \Delta_T \Leftrightarrow x \in T^\times$.

The last equivalence follows from $T_\sharp^\times \subseteq T^\times$: if $x^\sharp \in \Delta_T$, there is $t \in T^\times$ and $s \in T_\sharp^\times$

[3] Pointed out by the referee, improving an earlier version of Theorem 8.21.

such that $xs = t$, which implies $x = t/s \in T^{\times}$; the converse is obvious. As for the first equivalence, we have:

($\Rightarrow$) in (V) : The hypothesis $\mathfrak{e}^-(x) \in \mathcal{I}$ implies $H(\mathfrak{e}^-(x)) \cap Y_T = \emptyset$, and Fact 2 entails $A\mathfrak{e}^-(x) = T\mathfrak{e}^-(x)$, whence $-|x|\mathfrak{e}^-(x) \in (T\mathfrak{e}^-(x))^{\times}$. Since $|x|\mathfrak{e}^+(x) \in (T_\sharp\mathfrak{e}^+(x))^{\times}$ and $x = |x|(\mathfrak{e}^+(x) - \mathfrak{e}^-(x))$, 8.13.(a.$(ii)$) yields $x \in T^{\times}$, as needed.

($\Leftarrow$) in (V) : Assume $x \in T^{\times}$, and let $\alpha \in Y_T$. From $x = |x|(\mathfrak{e}^+(x) - \mathfrak{e}^-(x))$ being a unit in $T \subseteq \alpha$, we obtain

$$\pi_\alpha(x) \;=\; \pi_\alpha(|x|)\,[\pi_\alpha(\mathfrak{e}^+(x)) - \pi_\alpha(\mathfrak{e}^-(x))] \;>_\alpha\; 0,$$

and so, as $\pi_\alpha(|x|) >_\alpha 0$, we get $\pi_\alpha(\mathfrak{e}^+(x)) >_\alpha \pi_\alpha(\mathfrak{e}^-(x))$. Since $\pi_\alpha(e) \in \{0, 1\}$ for every $e \in B(A)$, the latter inequality entails $\pi_\alpha(\mathfrak{e}^-(x)) = 0$. Since α is arbitrary in Y_T, we have $H(\mathfrak{e}^-(x)) \cap Y_T = \emptyset$, i.e., $e^-(x) \in \mathcal{I}$, completing the verification of (V) and the proof. ■

Corollary 8.22. *Under the assumptions of* 8.21, *if, in addition,* $\langle A, T \rangle$ *has bounded inversion, then* $G_T(A) = G_{red}(A) = B(A)$.

Proof. Immediate from 8.21, using 8.15, 8.16 and 8.17. ■

Theorem 8.21, Corollary 10.13, p. 84, of [**KZ**] and Proposition 7.8 yield

Corollary 8.23. *Let* $\langle A, T_\sharp \rangle$ *be a reduced* f*-ring.*

a) Let S *be a multiplicative subset of* A *consisting of non zero-divisors. Let* $A^S := AS^{-1}$ *be the ring of fractions of* A *by* S *and set*

$$T_\sharp{}^S \;:=\; \left\{\frac{x}{s^2} \in A^S : x \in T_\sharp \text{ and } s \in S\right\}.$$

Then, $T_\sharp{}^S$ *is the unique partial ordering on* A^S *such that* $\langle A^S, T_\sharp{}^S \rangle$ *is an* f*-ring extension of* $\langle A, T_\sharp \rangle$.

b) If T *is a preorder of* A^S*, containing* $T_\sharp{}^S$*, then* A^S *is* T*-faithfully quadratic.*

c) The BIR hull of $\langle A, T_\sharp \rangle$, $\langle A^*, T_\sharp{}^* \rangle$ *(cf. Proposition* 7.4*), is an* f*-ring extension of* $\langle A, T_\sharp \rangle$*. In particular, if* T *is a preorder of* A^* *containing* $T_\sharp{}^*$*, then* A^* *is* T*-faithfully quadratic.*

d) (1) *The group units,* $\mathscr{G}^*_{A^*, T^*_\sharp}$*, of the real semigroup associated to* $\langle A^*, T_\sharp{}^* \rangle$ *is a reduced special group.*

(2) *The group of units,* $\mathscr{G}^*_{A, T_\sharp}$*, of the real semigroup associated to* $\langle A, T_\sharp \rangle$ *is a reduced special group.*

Proof. Item (a) is the statement of Corollary 10.13 of [**KZ**], while (b) follows immediately from (a) and Theorem 8.21.(a). For (c, it suffices to observe that the multiplicative set $1 + T_\sharp$ consists of non zero-divisors. Indeed, since A is $T_\sharp$-reduced (i.e., $\sqrt[T_\sharp]{0} = (0)$), if $s \in A$ and $t \in T_\sharp$, then $s(1 + t) = 0$ entails $s^2 + s^2t = 0$, and so, Proposition 4.14.3 yields $s \in \sqrt[T_\sharp]{0}$, whence $s = 0$, as needed.

d) Item (d.1) is immediate from Proposition 7.8. For (d.2), it is shown in Corollary 3.6, p. 50 of [**DM10**] that the group of units of the real semigroup associated to any p-ring is naturally isomorphic to the group of units of the real semigroup associated to its BIR hull, i.e., in the present case, $\mathscr{G}^*_{A^*, T^*_\sharp} \simeq \mathscr{G}^*_{A, T_\sharp}$, and (d.2) follows from (d.1). ■

The preceding results apply to rings of continuous real-valued functions. Before stating the pertinent results, we set down:

8.24. **Notation and Remarks.** Let X be a topological space and let $\mathbb{C}(X)$ be the $\mathbb{R}$-algebra of real-valued continuous functions on X. Let $B(X)$ be the Boolean algebra of clopens in X. It is well-known that $\mathbb{C}(X)$, with pointwise defined order, algebraic and lattice operations is an f-ring whose positive cone is precisely the squares (indeed, this ring is *Pythagorean*), constituting an important example of a ΣFR with weak bounded inversion.

If $f \in \mathbb{C}(X)$ and $r \in \mathbb{R}$, recall (cf. 4.10.(e)) that

$$[\![f < r]\!] = \{x \in X : f(x) < r\},$$

with similar definitions for $[\![f = r]\!]$, $[\![f > r]\!]$, $[\![f \geq r]\!]$ and $[\![f \leq r]\!]$.

a) If $f \in \mathbb{C}(X)^{\times}$, then $\{[\![f < 0]\!], [\![f > 0]\!]\}$ is a partition of X into disjoint clopens ($[\![f = 0]\!] = \emptyset$).

b) If $U \in B(X)$, let

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U; \\ 0 & \text{if } x \notin U, \end{cases}$$

be the characteristic map of U. Then, $\chi_U \in B(\mathbb{C}(X))$, i.e., is an idempotent in $\mathbb{C}(X)$. Moreover, it is straightforward that:

(1) For all $f \in \mathbb{C}(X)^{\times}$, $\mathfrak{e}^+(f)$ and $\mathfrak{e}^-(f)$ (8.13) are the idempotents corresponding to $[\![f > 0]\!]$ and $[\![f < 0]\!]$, respectively.

(2) $U \in B(X) \longmapsto \chi_U \in B(\mathbb{C}(X))$ is a Boolean algebra isomorphism.

c) The π-special group, $G(\mathbb{C}(X))$, associated to $\mathbb{C}(X)$ will be written

$$G(X) = \{\widehat{f} : f \in \mathbb{C}(X)^{\times}\}. \qquad \blacksquare$$

PROPOSITION 8.25. *Let X be a topological space. With notation as in* 8.24*,*

a) $\mathbb{C}(X)$ is faithfully quadratic and the map $\beta_X : G(X) \longrightarrow B(X)$, given by $\beta_X(\widehat{f})$ $= [\![f < 0]\!]$, is an isomorphism of reduced special groups.

b) The ring $\mathbb{C}(X)$ is completely faithfully quadratic. Moreover, if T is a proper preorder of $\mathbb{C}(X)$, the reduced special group $G_T(X)$ is a Boolean algebra.

Concerning transversal representation in $\mathbb{C}(X)$, we have

c) For all $f, g_1, \ldots, g_n \in \mathbb{C}(X)^{\times}$,

$$[\![f > 0]\!] \;\cap\; \textstyle\bigcap_{k=1}^{n} [\![g_k < 0]\!] \;=\; \emptyset \;=\; [\![f < 0]\!] \;\cap\; \textstyle\bigcap_{k=1}^{n} [\![g_k > 0]\!]$$

iff *$f \in D^t(g_1, \ldots, g_n)$; moreover, for each $k \in \underline{\boldsymbol{n}}$ there is $c \in (\mathbb{C}(X)^{\times})^2$ such that $u = f - cg_k \in \mathbb{C}(X)^{\times}$ and $[\![u > 0]\!] \cap \bigcap_{j \neq k} [\![g_j < 0]\!] = \emptyset = [\![u < 0]\!] \cap \bigcap_{j \neq k} [\![g_j > 0]\!]$.*

PROOF. a) Since $\mathbb{C}(X)$ is a Pythagorean f-ring, 8.20 implies its quadratic faithfulness. It follows straightforwardly from items (1) and (2) in 8.24.(b) that β_X is, by any other name, just $\mathfrak{b}_{\mathbb{C}(X)}$; hence, the stated RSG-isomorphism also follows from 8.20.

Item (b) is an immediate consequence of (a) and Theorem 8.21.

c) As noted in 8.24.(b.1), $\mathfrak{e}^+(f)$ and $\mathfrak{e}^-(f)$ are the idempotents corresponding to the clopens $[\![f > 0]\!]$ and $[\![f < 0]\!]$, respectively. Now the Boolean algebra isomorphism in 8.24.(b.2) shows that the hypotheses in (c) are just a rephrasing of those in Theorem 8.18. ■

For the notion of real holomorphy ring of a ring, the reader is referred to [**KZ**], section 2. The preceding results yield:

Corollary 8.26. *The real holomorphy ring of* $\mathbb{C}(X)$*, X a topological space, is completely faithfully quadratic.*

Proof. Without loss of generality, we may assume X is completely regular ([**GJ**], Thm. 3.9, p. 41). By Example 4.13, pp. 37-38 in [**KZ**], the real holomorphy ring of $\mathbb{C}(X)$ is naturally isomorphic to $\mathbb{C}_b(X)$, the ring of bounded real-valued continuous functions on X, which in turn, is ring isomorphic to $\mathbb{C}(\beta X)$, where βX is the Stone-Čech compactification of X (cf. [**GJ**], Theorem 6.5 and p. 88); it is straightforward that this isomorphism induces a bijection between the preorders of $\mathbb{C}_b(X)$ and those of $\mathbb{C}(\beta X)$ and the conclusion follows from Theorem 8.21.(a). ■

In the case of $\mathbb{C}(X)$, Corollary 8.23 has the following interesting consequence:

Corollary 8.27. *Let $S \subseteq \mathbb{C}(X)$ be a multiplicative set such that for all $f \in S$, the interior of $[\![f = 0]\!]$ is empty. Then, the ring of fractions, $\mathbb{C}(X)S^{-1}$, is completely faithfully quadratic.*

Proof. Note that all elements of S are non zero-divisors and that the natural partial order on $\mathbb{C}(X)S^{-1}$, described in 8.23.(a), is just squares. The conclusion then follows from 8.23.(b). ■

4. Unit-Reflecting Preorders and Rings of Continuous Functions

In this section we show that if X is a compact Hausdorff space, *any* proper preorder on $\mathbb{C}(X)$ is unit-reflecting (Theorem 8.29). This will be generalized by Theorem 9.13, but the latter's proof requires 8.29. As a preparatory step, we need

Lemma 8.28. *If X is a topological space and $K \neq \emptyset$ is a closed set in X, then $P_K = \{f \in \mathbb{C}(X) : f\restriction K \geq 0\}$ is a proper unit-reflecting preorder of $\mathbb{C}(X)$, that is of bounded inversion iff $K = X$.*

Proof. By Theorem 3.9, p. 41 of [**GJ**], there is no loss of generality in assuming X completely regular. Clearly, P_K is a proper preorder of $\mathbb{C}(X)$ and any unit in $\mathbb{C}(X)$, strictly positive on K, is a unit in P_K. Hence, by Lemmas 4.13.(c) and 8.5, P_K is unit-reflecting: via the identification of X with a subspace of the closed points of $\mathrm{Sper}(\mathbb{C}(X))$ by the map η of 4.13.(c), K is contained in the subspace of closed points of $\mathrm{Sper}(\mathbb{C}(X), P_K)$, satisfying condition (3) in 8.5.

By complete regularity, if $p \notin K$, there is $g \in \mathbb{C}(X)$ with $g(p) = -1$ and $g\restriction K = 0$. Thus, $g \in P_K$, but $1 + g \notin \mathbb{C}(X)^{\times}$; it follows that P_K has bounded inversion iff $K = X$. ■

Theorem 8.29. *If X is a compact Hausdorff space, then* any *proper preorder on $\mathbb{C}(X)$ is unit-reflecting.*

Proof. Let $A = \mathbb{C}(X)$; write Y^* for the compact Hausdorff space of closed points in $\mathrm{Sper}(A)$, and fix a proper preorder T of A. Observe that for each $f \in T$, the closed set $[\![f \geq 0]\!] = \{x \in X : f(x) \geq 0\}$ is non-empty: otherwise, $-f \in (A^2)^\times \subseteq T^\times$, hence $-1 \in T$, which is impossible. Define

$$K_T \;=\; \bigcap \{[\![f \geq 0]\!] : f \in T\}.$$

Note that $[\![f \geq 0]\!] = [\![f^- = 0]\!]$, where f^- is the *negative part* of f:

$$f^-(x) \;=\; \begin{cases} -f(x) & \text{if } f(x) \leq 0; \\ 0 & \text{if } f(x) > 0. \end{cases}$$

Since $f^- \geq 0$ on X, then $f^- \in A^2 \subseteq T$. Next, observe that if $g_1, \ldots, g_n$ are in T, then, since $\sum_{j=1}^n (g_j^-)^2 \in T$, we get

$$\bigcap_{i=1}^n [\![g_j \geq 0]\!] \;=\; \bigcap_{i=1}^n [\![g_j^- = 0]\!] \;=\; [\![\Sigma_{j=1}^n (g_j^-)^2 \;=\; 0)]\!],$$

a non-empty closed set in X. Thus, the family $\{[\![f \geq 0]\!] : f \in T\}$ has the finite intersection property, and compactness of X guarantees that K_T is a *non-empty* closed set. As in Lemma 8.28, let $P_{K_T} = \{f \in A : f\restriction K_T \geq 0\}$. Clearly, $T \subseteq P_{K_T}$. We have

FACT 8.30. *For all $\alpha \in Y^*$, $T \subseteq \alpha \;\Leftrightarrow\; P_{K_T} \subseteq \alpha$.*

PROOF. It suffices to check the implication $(\Rightarrow)$. By Lemma 4.13.(c), we may identify Y^* with X. Assume, towards a contradiction, that for some $x \in X$, there is $f \in P_{K_T}$, with $f(x) < 0$. Then, $x \notin K_T$ and so there is $t \in T$, such that $t(x) < 0$, contradicting the fact that t is non-negative in the maximal ordering, α_x, associated to x. □

By 8.28, P_{K_T} is a unit-reflecting preorder of A. Thus, by Fact 8.30, to show that T is unit-reflecting, it suffices to establish $P_{K_T}^\times = T^\times$, or equivalently, $P_{K_T}^\times \subseteq T^\times$. This will be a consequence of the following

FACT 8.31. *Let V be a clopen set in X, disjoint from K_T. Let $X \xrightarrow{\chi_V} \{0, 1\}$ be the characteristic map of V ($\chi_V(x) = 1 \;\Leftrightarrow\; x \in V$). Then, $-\chi_V \in T$, i.e., $\chi_V \in \mathrm{supp}(T)$.*

PROOF. Since χ_V is an idempotent in A, clearly $\chi_V \in T$. To show that $-\chi_V \in T$, observe that for each $p \in V$, there is $t \in T$ such that $t(p) < 0$. Hence, for each $p \in V$, there is a triple, $\langle U_p, t_p, \varepsilon_p \rangle$, where U_p is an open neighborhood of p *contained* in V, $\varepsilon_p > 0$, and such that:

$$p \in U_p \subseteq \{z \in X : t_p(z) < -\varepsilon_p\} \cap V.$$

Since V is compact, $\{U_p : p \in V\}$ has a finite subcovering; by reindexing, let $U_1, \ldots, U_n$ be this subcovering, and let $\langle U_k, t_k, \varepsilon_k \rangle$, $k \in \underline{\boldsymbol{n}}$, be the associated triples.

Let $\varphi_1, \ldots, \varphi_n$ be a partition of unity subordinate to $\{U_1, \ldots, U_n\}$ (cf. Problem 5.W, p. 171, in [**Ke**], or Theorem 5.1.9, p. 301, in [**En**]). Therefore:

- For each $k \in \underline{\boldsymbol{n}}$, φ_k is a continuous function with values in [0, 1] and support contained in U_k. In particular, $\varphi_k \in A^2 \subseteq T$ and $\varphi_k \chi_V = \varphi_k$;
- For all $p \in V$, $\sum_{k=1}^n \varphi_k(p) = 1$.

Let $h \;=\; \sum_{k=1}^n \varphi_k t_k \;=\; \sum_{k=1}^n \varphi_k t_k \chi_V$. Note that $h = h\chi_V$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$. We claim that $h \leq -\varepsilon$ on V. Indeed, for $p \in V$, let $C_p = \{k \in \underline{\boldsymbol{n}} : \varphi_k(p) > 0\}$; then $\sum_{k \in C_p} \varphi_k(p) = 1$ and $p \in \bigcap_{k \in C_p} U_k$. But then we obtain, recalling that $t_k < -\varepsilon_k$ on U_k, $k \in \underline{\boldsymbol{n}}$:

$$h(p) \;=\; \sum_{j=1}^{n} \varphi_j(p)t_j(p) \;=\; \sum_{k\in C_p} \varphi_k(p)t_k(p)$$
$$\leq\; \sum_{k\in C_p} \varphi_k(p)\,(-\varepsilon_k) \;\leq\; -\varepsilon \sum_{k\in C_p} \varphi_k(p) \;=\; -\varepsilon,$$

as claimed. Clearly, $h \in T$; since $h \leq -\varepsilon\chi_V$, we get $-\varepsilon\chi_V \in T$; scaling by $(1/\varepsilon)\chi_V \in A^2 \subseteq T$, yields $-\chi_V \in T$, establishing Fact 8.31. □

Now, let f be a unit in P_{K_T}. Then, $f = f^+ - f^-$, with $f^+ \in A^2 \subseteq T$ ($f^+ \geq 0$ on X). Since f is a unit, $[\![f \geq 0]\!]$ is equal to $[\![f > 0]\!]$ and so a clopen set in X, containing K_T. Moreover, $V := [\![f < 0]\!] = [\![f^- > 0]\!]$ is a clopen set, disjoint from K_T, with $\chi_V = f^-/|f|$ (recall: χ_V is the characteristic map of the clopen V). Hence, Fact 8.31 and $|f| \in T$ entail $-f^- = -\chi_V \cdot |f| \in T$, showing that $f = f^+ - f^- \in T$, as needed to conclude the proof of Theorem 8.29. ■

Example 8.32. **Example of a non-unit reflecting preorder.** Examples are known (see Stengle [**St**], Reznik [**R**]) of irreducible, positive semi-definite polynomials $f \in \mathbb{Z}[X, Y, Z]$ such that no odd power of f is a sum of squares in $\mathbb{R}[X, Y, Z]$:

$$f(X, Y, Z) \;=\; X^3Z^3 + (Y^2Z \;-\; X^3 \;-\; Z^2X)^2,$$

is such an example. We shall prove that in the ring

$$A \;=\; \mathbb{R}[X, Y, Z]\, f^{-1},$$

i.e., the ring of fractions of $\mathbb{R}[X, Y, Z]$ by the multiplicative set $\{f^n : n \in \mathbb{N}\}$, the preorder ΣA^2 *is not* unit-reflecting.

It is clear that $f \in A^{\times}$. By the stated property of f we have

Fact. $f \notin \Sigma A^2$.

Proof of Fact. Assume $f \in \Sigma A^2$, i.e.,

$$f \;=\; \sum_{i=1}^{n} \left(\frac{g_i}{f^{m_i}}\right)^2 \;=\; \frac{\Sigma g_i^2}{f^{2k}},$$

for suitable $g_i \in \mathbb{R}[X, Y, Z]$ and $k \in \mathbb{N}$. Then $f^{2k+1} = \Sigma\, g_i^2$ is a sum of squares in $\mathbb{R}[X, Y, Z]$, contrary to the choice of f. □

It remains to be proved:

(†) $\qquad f \in \bigcap \{\alpha \setminus \mathrm{supp}(\alpha) : \alpha \in \mathrm{Sper}(A)\}.$

Let $S = \{f^n : n \in \mathbb{N}\}$ and let $\iota_S : \mathbb{R}[X, Y, Z] \longrightarrow A$ be the natural embedding, $a \in A \mapsto \dfrac{a}{1}$. Then, identifying modulo the (injective) map $\mathrm{Sper}(\iota_S)$, we have $\mathrm{Sper}(A) \;=\; \{\alpha \in \mathrm{Sper}(\mathbb{R}[X,Y,Z]) : f \notin \mathrm{supp}(\alpha)\}$. Thus, (†) is equivalent to:

(††) $\qquad$ For all $\alpha \in \mathrm{Sper}(\mathbb{R}[X,Y,Z])$, $\;f \notin \mathrm{supp}(\alpha) \;\Rightarrow\; \pi_\alpha(f) >_\alpha 0.$

Proof of (††). Fix $\alpha \in \mathrm{Sper}(\mathbb{R}[X,Y,Z])$ such that $f \notin \mathrm{supp}(\alpha)$, i.e., $\pi_\alpha(f) \neq 0$.

Let $\overline{x} \in \mathbb{R}^3$ be the unique point such that $\alpha \subseteq \alpha_{\overline{x}}$ (= the maximal element of $\mathrm{Sper}(\mathbb{R}[X,Y,Z])$ over α). Note that $\pi_{\alpha_{\overline{x}}}(f) = f(x) \in \mathbb{R}$.

The inclusion $\alpha \subseteq \alpha_{\overline{x}}$ gives rise to a morphism of ordered rings,

$$\gamma \;=\; \gamma_{\alpha,\overline{x}} : \langle\, A_\alpha, \leq_\alpha \rangle \longrightarrow \langle\, A_{\alpha_{\overline{x}}}, \leq_{\alpha_{\overline{x}}} \rangle \;=\; \langle\, \mathbb{R}, \mathbb{R}^2 \,\rangle,$$

such that $\gamma \circ \pi_\alpha = \pi_{\alpha_{\overline{x}}}$. We consider two cases:

1) $f(\overline{x}) \neq 0$. Since f is positive semi-definite, $f(\overline{x}) > 0$ (in $\mathbb{R}$), and the fact that $\gamma(\pi_\alpha(f)) = f(\overline{x}) > 0$ entails $\pi_\alpha(f) >_\alpha 0$.

2) $f(\overline{x}) = 0$. Assume, towards a contradiction, that $\pi_\alpha(f) <_\alpha 0$. Since π_α is a ring morphism,

$$\pi_\alpha(f) \;=\; f(\pi_\alpha(X),\, \pi_\alpha(Y),\, \pi_\alpha(Z)) <_\alpha 0.$$

This means that in the ring A_α – and so also in its fraction field k_α –, the following statement (in the language for ordered rings) holds, with witness $\langle\, \pi_\alpha(X), \pi_\alpha(Y), \pi_\alpha(Z)\,\rangle$

(*) $$\langle\, k_\alpha, \leq_\alpha \rangle \models \exists\; v_1\; v_2\; v_3\; (f(v_1, v_2, v_3) \;<\; 0).$$

Hence, it also holds in the real closure $\overline{k_\alpha}$ of $\langle\, k_\alpha, \leq_\alpha \rangle$. Now, since f *has integer coefficients*, by completeness of the first-order theory of real closed fields, the sentence in (*) (without parameters!) holds in any other real closed field, e.g., the reals, $\mathbb{R}$. Thus, $\mathbb{R} \models \exists\; \overline{v}\; f(\overline{v}) < 0$, contradicting that f is positive semi-definite.

Therefore, (since $\pi_\alpha(f) \neq 0$), we obtain $\pi_\alpha(f) >_\alpha 0$ for all $\alpha \in \mathrm{Sper}(\mathbb{R}[X, Y, Z])$, proving (††) and concluding our example. ■

5. Some Applications and Examples

DEFINITION 8.33. *Let A be a ring.*

a) A is a **completely real function ring (CRFR)** *if A is a completely real ΣFR* (cf. 4.14.4.(b), 8.8.(c)).

b) A is a **weakly real closed ring (WRCR)** *if it verifies the following properties:*

[WRCR 1] : *A is reduced;*

[WRCR 2] : *A^2 is the positive cone of a partial order $\leq$ on A, with which it is a f-ring;*

[WRCR 3] : *For all a, $b \in A$, $\;0 \leq a \leq b \;\Rightarrow\; b$ divides a^2.* ■

REMARKS 8.34. a) The missing axiom for **real closed rings** in 8.33.(b), is:

For all prime ideals $\mathfrak{p} \subseteq A$, the field of fractions of $A/\mathfrak{p}$ is real closed and $A/\mathfrak{p}$ is integrally closed in it.

b) For more information on real closed rings, see [**Sc1**], [**Sc2**]; for a first order axiomatization, see [**PS**].

c) Clearly, all rings in Definition 8.33 are f-rings, with $T_\sharp = \Sigma A^2$. ■

The relation between these classes of rings are described in

LEMMA 8.35. *a) If $\langle\, A, T\,\rangle$ is a p-ring satisfying* [WRCR 3] *(in the order induced by T), then all radical ideals in A are real.*

b) Every CRFR is a ΣFA with weak bounded inversion, and every WRCR is a Pythagorean CRFR.

c) The properties of being a CRFR or a WRCR are preserved by localization at an idempotent.

PROOF. a) Let $\leq$ be the binary relation induced by the preorder T, i.e., $x \leq y$ iff $\;y - x \in T$. Let $\mathfrak{r}$ be a radical ideal in A; suppose $s = \sum_{j=1}^{n} a_j^2 \in \mathfrak{r}$ and fix

$1 \le j \le n$. Since $A^2 \subseteq T$, we get $0 \le a_j^2 \le s$ and [WRCR 3] yields $c \in A$ such that $sc = a_j^4$. Since, $s \in \mathfrak{r}$, we obtain $a_j^4 \in \mathfrak{r}$, entailing $a_j \in \mathfrak{r}$, as desired.

b) The first assertion is a special case of Lemma 4.15.(a). For the second, let A be a WRCR; since it is reduced, item (a) implies that it is completely real. Since A^2 is the positive cone of a ring-po on A, by Fact 4.17.(b), A is Pythagorean and so ΣA^2 is a partial order on A with which it is an f-ring and A is a CRFR, as claimed.

c) A straightforward modification of the proof of 8.10. We make only the following comments, where e is an idempotent in A :

• If A is a CRFR, by Theorem 4.16.(a) Ae is completely real and so, by Fact 4.17.(b), $\Sigma(Ae)^2$ is a ring-po on Ae.

• It is straightforward that Ae satisfies [WRCR 1] and [WRCR 2] in 8.33.(b). Now suppose $0 \le a \le b$, with $a, b \in Ae$; then these inequalities hold in A and there is $c \in A$ such that $cb = a^2$. Multiplying this equality by e yields $(ce)be = (ce)b = a^2e = (ae)^2 = a^2$, showing that Ae satisfies [WRCR 3]. ■

From Theorem 8.21 we obtain

Corollary 8.36. *Every ΣFR is completely faithfully quadratic. In particular, all ΣFAs, all WRCRs, all CRFRs and all real closed rings are completely faithfully quadratic.* ■

The examples that follow will show, in particular, that the inclusions WRCR $\subseteq$ CRFR $\subseteq$ ΣFA $\subseteq$ ΣFR are proper.

Remarks 8.37. If X is a completely regular space, then $\mathbb{C}(X)$ is a classical example of a *real closed ring* and so a WRCR. However, there are real closed rings which are not even elementarily equivalent to any one of this type, let alone isomorphic; see section 4 in [**Tr**].

If X is completely regular, then $\mathbb{C}_b(X)$ is also a real-closed ring, because there is an *order-preserving* ring isomorphism between $\mathbb{C}_b(X)$ and $\mathbb{C}(\beta X)$, where βX is the Stone-Čech compactification of X (cf. Theorem 6.5, p. 86 and Remark 6.6.(b), p. 88 in [**GJ**]).

Even without invoking these general results, a minimal acquaintance with rings of continuous functions will allow the reader to give direct, elementary proofs that the rings $\mathbb{C}(X)$ and $\mathbb{C}_b(X)$ are WRCRs, for all X; see, e.g., [**GJ**], 5.5, for an argument proving [WRCR 3]. ■

Example 8.38. (Example of a CRFR that is not a WRCR.) Let X be a topological space and let $A = \mathbb{C}(X, \mathbb{Q})$ be the ring of locally constant rational-valued functions on X. Endowed with the pointwise partial order, A is a lattice-ordered ring. Note that for $f \in A$,

$$f \ge 0 \;\;\Leftrightarrow\;\; \forall\, x \in X,\;\; f(x) \in \Sigma\mathbb{Q}^2 \;\;\Leftrightarrow\;\; f \in \Sigma A^2,$$

where the last equivalence follows from f being locally constant. Hence, $\langle\, A, \Sigma A^2 \,\rangle$ is a real f-ring (note: $\mathfrak{m}_x = \{f \in A : f(x) = 0\}$ is a real maximal ideal, and $\bigcap_{x \in X} \mathfrak{m}_x = (0)$). To show that A is completely real, we first check it satisfies [WRCR 3]. Indeed, if $0 \le f \le g$, then $[\![g = 0]\!] \subseteq [\![f = 0]\!]$ are both **clopen** in X and it is straightforward that the map $h : X \longrightarrow \mathbb{Q}$ defined by

$$h(x) \;=\; \begin{cases} f/g & \text{if } g(x) \neq 0; \\ 0 & \text{if } x \in [\![g = 0]\!]. \end{cases}$$

is locally constant on X, that is, $h \in A$. Hence, $hf \in A$, with $f^2 = hfg$. Now, the argument of 8.35.(a) shows that every prime ideal in A is real. Since A *is not* Pythagorean, it furnishes an example of a CRFR (and thus of a ΣFA with weak bounded inversion) that is not a WRCR. Moreover, in general, $\langle\, A, \Sigma A^2 \,\rangle$ *is not Archimedean* (cf. definition in 9.3): just consider $\mathbb{Q}^{\mathbb{N}} = \mathbb{C}(\mathbb{N}, \mathbb{Q})$, where $\mathbb{N}$ has the discrete topology, and any strictly decreasing sequence converging to 0. ■

EXAMPLE 8.39. (Examples of ΣFRs that are neither $\mathbb{Q}$-algebras, nor CRFRs.) Let

$$D = \{m/2^n \in \mathbb{Q} : m \in \mathbb{Z} \text{ and } n \geq 0\}$$

be the ring of dyadic rationals. D is endowed with the linear order induced by $\mathbb{Q}$. Clearly D is a real integral domain, $2 \in D^{\times}$ and $x \geq 0$ iff $x \in \Sigma D^2$. Thus, D is a ΣFR and not a $\mathbb{Q}$-algebra.

Let $A = \mathbb{C}(X, D)$ be the ring of locally constant D-valued functions on a topological space X. Clearly, $2 \in A^{\times}$. Since A is a lattice-ordered subring of D^X and the restriction of the coordinate projections of D^X to A are onto D, A is a reduced f-ring (cf. Fact 8.3.(c)). For $x \in X$, set $\mathfrak{p}_x = \{f \in A : f(x) = 0\}$; then $A/\mathfrak{p}_x \simeq D$, and so $\mathfrak{p}_x$ is a real prime ideal; obviously, $\bigcap_{x \in X} \mathfrak{p}_x = (0)$; hence, A is real. Since every positive integer is a sum of four squares (Lagrange's Theorem), the pointwise partial order on A coincides with ΣA^2, and A is a ΣFR and hence completely faithfully quadratic. However, A is not a $\mathbb{Q}$-algebra and does not have weak bounded inversion. Moreover, A *is not a CRFR* : if $pD = \{m/2^n : m \in p\mathbb{Z},\, n \geq 0\}$, where $p \neq 2$ is a prime in $\mathbb{Z}$, then, fixing $x \in X$ and letting $\mathfrak{m} = \{f \in A : f(x) \in pD\}$, we get $A/\mathfrak{m} = D/pD = \mathbb{Z}_p$; hence, $\mathfrak{m}$ is a non-real maximal ideal. ■

CHAPTER 9

Strictly Representable Rings

In this Chapter we first show that if $\langle A, T \rangle$ is a p-ring where T is an Archimedean preorder with bounded inversion, then $\langle A, T \rangle$ is T-faithfully quadratic (Theorem 9.9). An interesting consequence is that the real holomorphy ring of any formally real field is Σ-faithfully quadratic. We then show that if $\langle A, T \rangle$ is an Archimedean p-ring with bounded inversion and P is a preorder of A containing T, then P is unit-reflecting and $\langle A, P \rangle$ is P-faithfully quadratic (Theorem 9.13).

9.1. **Notation.** Let X be a compact Hausdorff space.

a) If $f \in \mathbb{C}(X)$, let $\|f\|_\infty = \max_{x \in X} |f(x)|$ be the sup or uniform convergence norm on $\mathbb{C}(X)$. Recall that for $f, g \in \mathbb{C}(X)$, we have $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$.

b) Write D^v_X and D^t_X for value representation and transversal representation in $\mathbb{C}(X)$.

c) Notation $[\![f \geq 0]\!]$, $[\![f > 0]\!]$, etc. ($f \in \mathbb{C}(X)$) is defined in 4.10.(e). ■

Definition 9.2. *Let X be a compact Hausdorff space and let $\langle A, T \rangle$ be p-ring. We say that $\langle A, T \rangle$ is* **strictly representable in $\mathbb{C}(X)$ mod T**, *or simply* **strictly representable in X**, *if there is a ring morphism, $\varphi : A \longrightarrow \mathbb{C}(X)$, such that*

(1) *$\varphi[A]$ is dense in $\mathbb{C}(X)$ for the topology of the sup norm.*

(2) $\varphi[T] \subseteq \{f \in \mathbb{C}(X) : [\![f \geq 0]\!] = X\}$.

(3) *For all $a \in A$, $[\![\varphi(a) > 0]\!] = X \Leftrightarrow a \in T^\times$.*

The morphism φ is called a **strict representation** *of $\langle A, T \rangle$ in X.* ■

Before characterizing strictly representable rings, we register the following

Remark 9.3. Let $\langle A, T \rangle$ be a p-ring. Recall that T is **Archimedean** if for all $a \in A$ there is $n \in \mathbb{N}$ such that $n - a \in T$. References on this theme are [**BS**], [**Be3**] and [**Pr**]; the latter (p. 91) contains an exposition of the history and ideas behind the Becker-Schwartz version ([**BS**]) of the Kadison-Dubois Theorem (cf. also [**Kr**]), stated here in an equivalent, but slightly different form:

9.4. Theorem ([**BS**]) *If $\langle A, T \rangle$ is Archimedean, there is a compact Hausdorff space X_T and a ring morphism, $\varphi : A \longrightarrow \mathbb{C}(X_T)$, such that*

(1) *$\mathbb{Q} \cdot \varphi[A]$ is dense in $C(X_T)$ with respect to the sup norm;*

(2) $\varphi[T] \subseteq \{f \in \mathbb{C}(X_T) : [\![f \geq 0]\!] = X_T\}$;

(3) *For all $a \in A$, $[\![\varphi(a) \geq 0]\!] = X_T \Rightarrow na + 1 \in T$, for all $n \in \mathbb{N}$.* □

Note that if the ring A in 9.4 is a $\mathbb{Q}$-algebra − as is the case of any p-ring with bounded inversion − then 9.4(1) is equivalent to $\varphi[A]$ being dense in $\mathbb{C}(X_T)$. ■

Since this will be needed later, we succinctly describe the space X_T and the associated representation of $\langle A, T \rangle$ in $\mathbb{C}(X_T)$. The reader may also consult [**Pr**], [**BS**]. Chapter 4 of [**Be2**] (unpublished) contains a generalization of 9.4 to T-modules.

9.5. **Constructions**

(1) Given an Archimedean preorder, $\langle A, T \rangle$, let

$$X_T \;=\; \text{the set of p-ring morphisms from } \langle A, T \rangle \text{ to } \langle \mathbb{R}, \mathbb{R}^2 \rangle.$$

(2) For each $a \in A$, let $ev(a) : X_T \longrightarrow \mathbb{R}$ be the evaluation map

$$(ev) \qquad ev(a)(h) = h(a). \qquad (h \in X_T)$$

(3) We endow X_T with the weak topology induced by $\{ev(a) : a \in A\}$, i.e. the coarsest topology on X_T that makes all these maps continuous.

(4) The map $\varphi = ev : A \longrightarrow \mathbb{C}(X_T)$ described in (ev) above is the ring morphism with the properties in the statement of Theorem 9.4. ■

Let $Y_T = \mathrm{Sper}(A, T)$ and Y_T^* be the compact Hausdorff space of the closed points in Y_T (cf. 4.12 and 4.13.(b)). Although indicated in [**Pr**], we include a sketch of the natural homeomorphism between Y_T^* and X_T, which will be explicitly used later on.

PROPOSITION 9.6. *Let T be an Archimedean preorder of A. The spaces X_T and Y_T^* are homeomorphic via the map*

$$\mathfrak{h} : h \in X_T \;\longmapsto\; \alpha_h \;=\; h^{-1}[\mathbb{R}^2] \in Y_T^*.$$

Its inverse is given by the correspondence

$$\mathfrak{g} : \beta \in Y_T^* \longmapsto \iota_\beta \circ \pi_\beta \ (\in X_T),$$

where, for $\alpha \in Y_T$, $\pi_\alpha : A \longrightarrow k_\alpha$ is the canonical ring morphism and ι_β is the order embedding of the field $\langle k_\beta, \leq_\beta \rangle$ into $\langle \mathbb{R}, \mathbb{R}^2 \rangle$ (see proof below). In particular, X_T is a compact Hausdorff space.

PROOF. (Sketch) The first assertion is proved in [**Be2**], Proposition 4.1.9 and Remark 4.1.10., p. 135.

Concerning the inverse map $\mathfrak{g}$, observe first that the assumption "T Archimedean" entails that the linear order of the ring $A_\alpha = A/\mathrm{supp}(\alpha)$ (induced by α) is also Archimedean, for all $\alpha \in Y_T = \mathrm{Sper}(A, T)$; note A_α may contain (non-invertible) infinitesimals. However, Proposition 4.1.9 in [**Be2**] shows that, whenever $\beta \in Y_T^*$, the ordered *field* $\langle k_\beta, \leq_\beta \rangle$ is Archimedean, and therefore admits a *unique* embedding into $\langle \mathbb{R}, \mathbb{R}^2 \rangle$ (Hahn Embedding Theorem). However, we only need the well-known

Fact. *For every $\beta \in Y_T^*$, the* unique *embedding of ordered fields, $\iota_\beta : \langle k_\beta, \leq_\beta \rangle \longrightarrow \langle \mathbb{R}, \mathbb{R}^2 \rangle$, satisfies $\beta = (\mathfrak{g}(\beta))^{-1}[\mathbb{R}^2]$ (where $\mathfrak{g}(\beta) = \iota_\beta \circ \pi_\beta$).* ■

Thus, for $\beta \in Y_T^*$, the map $\mathfrak{g}(\beta)$ is a p-ring morphism from A into $\mathbb{R}$, that is, $\mathfrak{g}(\beta) \in X_T$.

Now we check that $\mathfrak{h}$ and $\mathfrak{g}$ are mutually inverse maps:

• $(\mathfrak{h} \circ \mathfrak{g})(\beta) = \beta$, for $\beta \in Y_T^*$. This follows from the equality in the Fact, since $\mathfrak{h}(\mathfrak{g}(\beta)) = (\mathfrak{g}(\beta))^{-1}[\mathbb{R}^2] = \beta$.

• $(\mathfrak{g} \circ \mathfrak{h})(h) = h$, for $h \in X_T$. This follows from the uniqueness assertion in the Fact: given $h \in X_T$, set $\beta = \alpha_h \in Y_T^*$, and notice that both h and $\iota_{\alpha_h} \circ \pi_{\alpha_h}$ are p-ring morphisms from $\langle A, T \rangle$ to $\langle \mathbb{R}, \mathbb{R}^2 \rangle$, with $\alpha_h = h^{-1}[\mathbb{R}^2]$ $= (\iota_{\alpha_h} \circ \pi_{\alpha_h})^{-1}[\mathbb{R}^2]$ (see Fact above). Hence, uniqueness yields $h = \iota_{\alpha_h} \circ \pi_{\alpha_h}$, as required to prove the assertion. ∎

THEOREM 9.7. *For a p-ring $\langle A, T \rangle$ the following are equivalent:*

(1) *$\langle A, T \rangle$ is strictly representable in some compact Hausdorff space;*

(2) *$\langle A, T \rangle$ is an Archimedean p-ring with bounded inversion.*

Moreover, if $\varphi : \langle A, T \rangle \longrightarrow \mathbb{C}(X)$ is a strict representation of $\langle A, T \rangle$ and these equivalent conditions are satisfied, then

(†) *For all $a \in A$, $\varphi(a) \in \mathbb{C}(X)^\times \;\Leftrightarrow\; a \in A^\times$.*

PROOF. (1) ⇒ (2) and (†): Let $\varphi : A \longrightarrow \mathbb{C}(X)$ be a strict representation of A in the compact Hausdorff space X. For $a \in A$, the function $\varphi(a)$ is bounded on X, i.e., for some $n \in \mathbb{N}$, $n - \varphi(a) = \varphi(n - a) > 0$ on all of X. By 9.2(3), $n - a \in T^\times$ $\subseteq T$, and T is indeed an Archimedean preorder of A. Next, if $t \in T$, then $\varphi(1 + t)$ $= 1 + \varphi(t)$ is, by 9.2(2), strictly positive on X and so a unit in $\mathbb{C}(X)^2$. But then 9.2(3) implies $1 + t \in T^\times$. Regarding (†), since φ is a ring morphism, it suffices to prove (⇒). If $\varphi(a) \in \mathbb{C}(X)^\times$, then $\varphi(a^2) = \varphi(a)^2$ is a unit in $\mathbb{C}(X)^2$ and another application of 9.2(3) entails $a^2 \in T^\times$, whence $a \in A^\times$, as desired.

(2) ⇒ (1): In view of 9.4, it only remains to show:

Fact. *If $\langle A, T \rangle$ is an Archimedean p-ring with bounded inversion and $\varphi : A \longrightarrow \mathbb{C}(X_T)$ is the representation of A in 9.4, then for all $a \in A$, $[\![\varphi(a) > 0]\!] = X_T \;\Leftrightarrow\; a \in T^\times$.*

Proof of Fact. To ease notation, write $\breve{a}$ for $\varphi(a)$, $a \in A$, and X for X_T. Implication (⇐) is clear: since a is a unit and φ is a ring morphism, $\breve{a} \in \mathbb{C}(X)^\times$, with $[\![\breve{a} \geq 0]\!]$ $= X$ (by 9.4(2)). Since $\breve{a}$ does not have zeros, we get $[\![\breve{a} > 0]\!] = [\![\breve{a} \geq 0]\!] = X$.

For the converse, first note (as observed in (ii'), page 92 of [**Pr**]) that 9.4(3) is equivalent to

9.4.(3′) For all $a \in A$, $[\![\breve{a} > 0]\!] = X \;\Rightarrow\; a \in T$.

9.4(3) ⇒ 9.4(3′): Suppose $[\![\breve{a} > 0]\!] = X$. By compactness, there is $n \geq 1$ so that $\breve{a} > \dfrac{1}{n}$ on all of X. Thus, $[\![\varphi(na - 1) > 0]\!] = X$ and 9.4(3) entails $na =$ $1 + (na - 1) \in T$. Since $n \in T^\times$, we get $a \in T$.

9.4(3′) ⇒ 9.4(3): If $[\![\breve{a} \geq 0]\!] = X$, then $\breve{a} > \dfrac{-1}{n}$ for all $n \in \mathbb{N}$, and $\varphi(na + 1) > 0$ on all of X. Hence, 9.4(3′) entails $na + 1 \in T$, for all $n \in \mathbb{N}$.

To complete the proof, assume $[\![\breve{a} > 0]\!] = X$. By 9.4(3′), $a \in T$, whence $\overline{a} \geq_\alpha 0$ for all $\alpha \in Y_T = \mathrm{Sper}(A, T)$. If $\overline{a}(\alpha) = 0$, i.e., $\pi_\alpha(a) = 0$, for some $\alpha \in Y_T$, choose $\beta \in Y_T^*$ such that $\alpha \subseteq \beta$. Using the morphism $\mathfrak{h}$ of Proposition 9.6, realize β as α_h, for some $h \in X = X_T$. The inclusion $\alpha \subseteq \beta$ gives $\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\beta)$, and hence

$a \in supp(\beta)$, i.e., $\pi_\beta(a) = 0$. From $\beta = \alpha_h$ (and $h = \iota_\beta \circ \pi_\beta$) follows $h(a) = 0$, i.e., $\breve{a}(h) = 0$, contradicting the assumption that $\breve{a} > 0$ on all of X.

Thus, $\overline{a}(\alpha) = 1$, for all $\alpha \in Y_T$. Now we invoke Theorem 4.11.(b) to obtain $(1 + s)a = 1 + t$, for some $s, t \in T$. Since $\langle A, T \rangle$ has bounded inversion, $1 + s$, $1 + t \in A^\times$, whence, $a = (1 + t)/(1 + s) \in T^\times$, as asserted. ∎

Before proving that any Archimedean BIR, $\langle A, T \rangle$, is T-faithfully quadratic (Theorem 9.9), we present interesting examples of this class of rings, including the real holomorphy ring of a formally real field. Although the result that follows is known (cf. [**Be3**], [**KZ**], [**Be2**]), we include a brief comment on its item (b).

PROPOSITION 9.8. *a) If $\langle A, T \rangle$ is a p-ring, let*

$$A(T) \;=\; \{a \in A : \exists\, n \in \mathbb{N} \text{ such that } n \pm a \in T\},$$

be the convex hull of $\mathbb{Z}$ with respect to the preorder T of A. Then,

(1) *$A(T)$ is a subring of A and $T \cap A(T)$ is an Archimedean preorder of $A(T)$.*

(2) *If $t \in T$ and $1 + t \in T^\times$, then $1/(1+t) \in A(T)$. In particular, if $\langle A, T \rangle$ has bounded inversion, $\langle A(T), T \cap A(T) \rangle$ is an Archimedean BIR.*

b) If K is a formally real field and $H(K)$ is its real holomorphy ring, then $\langle H(K), \Sigma H(K)^2 \rangle$ is an Archimedean p-ring with weak bounded inversion.

PROOF. b) By definition, (see [**Be3**], p. 21)

$$H(K) \;=\; \{a \in K : \exists\, n \in \mathbb{N} \text{ such that } n \pm a \in \Sigma K^2\}.$$

Then, (b) follows from (a) upon observing that $\Sigma H(K)^2 = \Sigma K^2 \cap H(K)$. This, in turn easily follows from:

• $v(\Sigma_{i=1}^n a_i^2) = 2 \cdot \min_{1 \le i \le n}\{v(a_i)\}$, for any valuation v of K with formally real residue field ;

• $H(K)$ is the intersection of all valuations rings of K with residue field contained in $\mathbb{R}$ (cf. [**Be3**], [**Mar**]). ∎

THEOREM 9.9. *Every Archimedean p-ring with bounded inversion, $\langle A, T \rangle$, is T-faithfully quadratic.*

PROOF. Since $\langle A, T \rangle$ has bounded inversion, Theorem 7.6.(d) guarantees that value representation in $\langle A, T \rangle$ satisfies transversality for forms of arbitrary dimension, that is, if θ is a form over $A^\times$, then $D^t_{A,T}(\theta) = D^v_{A,T}(\theta)$. In particular, A verifies [T-FQ 1], although this will also be a consequence of the proof below.

Let $\varphi : \langle A, T \rangle \longrightarrow \mathbb{C}(X)$ be a representation of $\langle A, T \rangle$ (Theorem 9.7). Write $\breve{a}$ for $\varphi(a)$ and $\breve{A}$ for $\varphi[A]$. If $\psi = \langle c_1, \ldots, c_n \rangle$ is a form over $A^\times$, write $\breve{\psi}$ for $\langle \breve{c}_1, \ldots, \breve{c}_n \rangle$, the corresponding form over $\mathbb{C}(X)^\times$.

We first register the following

FACT 9.10. *If $a, b_1, \ldots, b_n \in A^\times$ and $\theta = \langle b_1, \ldots, b_n \rangle$, then*

$$\left\{ \begin{array}{c} [\![\breve{a} > 0]\!] \;\cap\; \bigcap_{j=1}^n [\![\breve{b}_j < 0]\!] \;=\; \emptyset \;=\; [\![\breve{a} < 0]\!] \;\cap\; \bigcap_{j=1}^n [\![\breve{b}_j > 0]\!] \\ \Downarrow \\ a \in D^t_{A,T}(\theta). \end{array} \right.$$

Proof. By Proposition 8.25.(b), $\breve{a} \in D_X^t(\breve{\theta})$, whence there are $f_1, \ldots, f_n \in (\mathbb{C}(X)^2)^\times$ such that

(I) $$\breve{a} \;=\; f_1\breve{b}_1 + \ldots + f_n\breve{b}_n.$$

Note that:

- For each $j \in \underline{\boldsymbol{n}}$, compactness and $X = [\![f_j > 0]\!]$ imply
$$m_j = \min\{f_j(p) \in \mathbb{R} : p \in X\} > 0 \quad \text{and} \quad \|\breve{b}_j\|_\infty > 0;$$
- Since $\breve{a} \in \mathbb{C}(X)^\times$, $[\![|\breve{a}| > 0]\!] = X$ and so
$$m \;=\; \min\{|\breve{a}(p)| \in \mathbb{R} : p \in X\} \;>\; 0.$$

Let $\varepsilon \;=\; \min\{m, m_1, \ldots, m_n\}$. Since $\breve{A}$ is a dense subalgebra of $\mathbb{C}(X)$ in the topology of uniform convergence (9.2.(1)),

(II) For each $j \in \underline{\boldsymbol{n}}$, there is $c_j \in A$ such that $\|\breve{c}_j \;-\; f_j\|_\infty < \dfrac{\varepsilon}{2n(1 + \|\breve{b}_j\|_\infty)}$.

(i) *For* $j \in \underline{\boldsymbol{n}}$, $c_j \in T^\times$. Indeed, fix $j \in \underline{\boldsymbol{n}}$; since $0 \;<\; \varepsilon \;\leq\; m_j = \min_{p \in X}\{f_j(p)\}$, (II) implies $|\breve{c}_j(p) - f_j(p)| \;<\; \dfrac{m_j}{2}$, for all $p \in X$; hence, $\breve{c}_j(p) > \dfrac{f_j(p)}{2} > 0$, and $\breve{c}_j \in (\mathbb{C}(X)^2)^\times$. Now, property (3) in Definition 9.2 yields $c_j \in T^\times$, as needed.

Define

(*) $$z \;=\; c_1b_1 + \ldots + c_nb_n.$$

Then,

(ii) $az \in T^\times$. We have $\breve{z} \;=\; \breve{c}_1\breve{b}_1 + \ldots + \breve{c}_n\breve{b}_n$; (I) and (II) entail

(III) $$\begin{cases} \|\breve{a} - \breve{z}\|_\infty &= \|(f_1 - \breve{c}_1)\breve{b}_1 \;+ \ldots +\; (f_n - \breve{c}_n)\breve{b}_n\|_\infty \\ &\leq \sum_{j=1}^n \|f_j - \breve{c}_j\|_\infty \, \|\breve{b}_j\|_\infty \\ &< \sum_{j=1}^n \dfrac{\varepsilon}{2n(1 + \|\breve{b}_j\|_\infty)} \, \|\breve{b}_j\|_\infty \;<\; \dfrac{\varepsilon}{2}. \end{cases}$$

Since $\dfrac{\varepsilon}{2} \leq \dfrac{m}{2} = \dfrac{\min_{p \in X} \; |\breve{a}(p)|}{2}$, (III) yields

(IV) $$[\![(\breve{az}) > 0]\!] \;=\; [\![\breve{a}\,\breve{z} > 0]\!] \;=\; X.$$

Indeed, for $p \in X$, (III) implies $|\breve{a}(p) - \breve{z}(p)| < \dfrac{\varepsilon}{2} \leq \dfrac{m}{2}$, that is,
$$\breve{a}(p) - \frac{m}{2} \;<\; \breve{z}(p) \;<\; \breve{a}(p) + \frac{m}{2}.$$

Hence, recalling that $X \;=\; [\![\breve{a} > 0]\!] \cup [\![\breve{a} < 0]\!]$:

- If $\breve{a}(p) > 0$, then $\breve{z}(p) \;>\; \breve{a}(p) - \dfrac{m}{2} \;\geq\; \breve{a}(p) - \dfrac{\breve{a}(p)}{2} \;=\; \dfrac{\breve{a}(p)}{2} > 0$;
- If $\breve{a}(p) < 0$, then $\breve{z}(p) \;<\; \breve{a}(p) + \dfrac{m}{2} \;\leq\; \breve{a}(p) + \dfrac{|\breve{a}(p)|}{2} \;=\; \dfrac{\breve{a}(p)}{2} < 0$,

establishing (IV). Now, (IV) and property (3) in 9.2 guarantee that $az \in T^\times$, as desired. From ***(ii)*** we get $z \in A^\times$ and $\dfrac{a}{z} = \dfrac{az}{z^2} \in T^\times$.

But then, ***(i)*** and (*) entail $z \in D^t_{A,T}(\theta)$ and so $a = z\,\dfrac{a}{z} \in D^t_{A,T}(\theta)$, establishing Fact 9.10. □

***A* verifies** [T-FQ 2] By Lemma 2.26.(c) it suffices to check that for every form $\theta = \langle b_1, \ldots, b_n \rangle$ over $A^\times$, $D^v_{A,T}(\theta) \subseteq \mathfrak{D}_{A,T}(\theta)$. Since $D^v_{A,T}(\theta) = D^t_{A,T}(\theta)$, pick c_i and z as in the proof of Fact 9.10. Hence, $z = c_k b_k + \sum_{i \neq k} c_i b_i$. Since, $\dfrac{a}{z} \in T^\times$, $a = \dfrac{a}{z} \cdot z = \dfrac{a}{z} c_k b_k + \sum_{i \neq k} \dfrac{a}{z} c_i b_i$, with the second summand in $D^v_{A,T}(b_1, \ldots, \check{b}_k, \ldots, b_n)$, as required by [T-FQ 2].

The satisfaction of [T-FQ 3] in $\langle A, T \rangle$ will be a consequence of Proposition 3.12 and the following

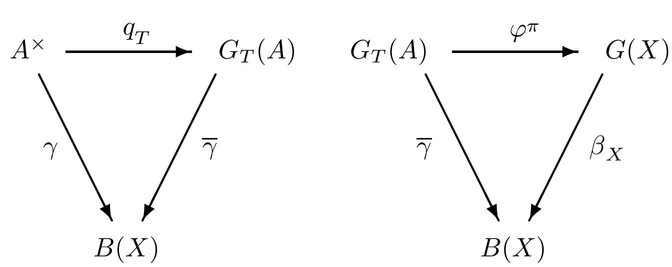
Fact 9.11. *Let $B(X)$ be the Boolean algebra of clopens in X. Then, with notation as above, in 8.24.(c) and in 8.25:*

a) If $u, w \in A^\times$, then $u \in D^v_{A,T}(1, w) \Leftrightarrow [\![\breve{u} < 0]\!] \subseteq [\![\breve{w} < 0]\!]$.

b) Let $\gamma : A^\times \longrightarrow B(X)$ be defined by $\gamma(u) = [\![\breve{u} < 0]\!]$. Then:

(1) *γ is a* surjective *group homomorphism, taking -1 to X, whose kernel is $T^\times$.*

(2) *γ factors uniquely through $G_T(A)$, yielding an isomorphism of π-special groups, $\overline{\gamma} : G_T(A) \longrightarrow B(X)$, making commutative the diagrams below, where $q_T : A^\times \longrightarrow G_T(A) = A^\times/T^\times$ is the canonical quotient morphism. Hence, φ^π, the π-RSG morphism induced by the representation φ of A in $\mathbb{C}(X)$, is also an isomorphism.*



Proof. a) ($\Rightarrow$) If $u = t + sw$, with $s, t \in T$, then $\breve{u} = \breve{t} + \breve{s}\,\breve{w}$, with $[\![\breve{s} \geq 0]\!] = [\![\breve{t} \geq 0]\!] = X$, which immediately entails $[\![\breve{u} < 0]\!] \subseteq [\![\breve{w} < 0]\!]$. Conversely, if this relation holds then, since $[\![1 > 0]\!] = X$ and $[\![1 < 0]\!] = \emptyset$, we obtain

$$\begin{aligned}[\![\breve{u} < 0]\!] \cap ([\![\breve{w} > 0]\!] \cap [\![1 > 0]\!]) &= [\![\breve{u} < 0]\!] \cap [\![\breve{w} > 0]\!] = \emptyset \\ &= [\![\breve{u} > 0]\!] \cap ([\![\breve{w} < 0]\!] \cap [\![1 < 0]\!]),\end{aligned}$$

whence Fact 9.10 yields $u \in D^t_{A,T}(1, w) \subseteq D^v_{A,T}(1, w)$, as needed.

b) (1) Note that $\gamma = \beta_X \circ can \circ (\varphi \restriction A^\times)$, where $\varphi : A \longrightarrow \mathbb{C}(X)$ is the representation map, *can* is the quotient map $\mathbb{C}(X)^\times \longrightarrow G(X) = \mathbb{C}(X)^\times/\mathbb{C}(X)^{\times 2}$ and $\beta_X : G(X) \longrightarrow B(X)$ is the π-RSG isomorphism defined in Proposition 8.25. This shows that γ is a group morphism, taking -1 in $A^\times$ to $X \in B(X)$. For $u \in A^\times$, we have $\gamma(u) = \emptyset \Leftrightarrow X = [\![\breve{u} > 0]\!]$, and property (3) in Definition 9.2 ensures that $u \in T^\times$, i.e., ker $\gamma = T^\times$.

It remains to check that γ is onto $B(X)$. For $V \in B(X)$, let $\chi_V : X \longrightarrow \mathbb{R}$ be given by

$$\chi_V(p) \;=\; \begin{cases} -1 & \text{if } p \in V; \\ 1 & \text{if } p \notin V. \end{cases}$$

Then, $\chi_V \in \mathbb{C}(X)^\times$ and the density of $\breve{A}$ in $\mathbb{C}(X)$ yields $v \in A$ such that $\|\breve{v} - \chi_V\|_\infty < \frac{1}{2}$, whence $[\![\breve{v} < 0]\!] = V$, $[\![\breve{v} > 0]\!] = X \setminus V$; thus $\breve{v} \in \mathbb{C}(X)^\times$, and (†) in 9.7 implies $v \in A^\times$, with $\gamma(v) = V$, as needed.

(2) Clearly, there is a unique bijective group morphism,

$$\overline{\gamma} : G_T(A) \longrightarrow B(X),$$

making the left-displayed diagram commute, and $\overline{\gamma}(-1) = X$. It follows immediately from Remark 8.12.(b) that, for $u, w \in A^\times$,

$$u \in D^v_{A,T}(1, w) \;\Leftrightarrow\; [\![\breve{u} < 0]\!] \subseteq [\![\breve{v} < 0]\!] \;\Leftrightarrow\; \gamma(u) \in D_{B(X)}(1, \gamma(w)),$$

and $\overline{\gamma}$ is an isomorphism of π-RSGs. It is clear from the construction that the right-displayed diagram is also commutative, as claimed. □

<u>A verifies [T-FQ 3] :</u> Since $\varphi : \langle A, T \rangle \longrightarrow \langle \mathbb{C}(X), \mathbb{C}(X)^2 \rangle$ is a p-ring morphism, $\mathbb{C}(X)$ is faithfully quadratic (Corollary 8.25.(a)), and $\varphi^\pi : G_T(A) \longrightarrow G(X)$ is an isomorphism of RSGs (by 9.11.(b)), Proposition 3.12 entails $\langle A, T \rangle \models$ [T-FQ 3], completing the proof. ■

Theorem 9.9 and Proposition 9.8.(b) yield

Corollary 9.12. *The real holomorphy ring of any formally real field is Σ-faithfully quadratic.* ■

We now generalize Theorem 8.29

Theorem 9.13. *Let $\langle A, P \rangle$ be an Archimedean BIR and let T be a preorder of A containing P. Then,*

a) T is unit-reflecting.

b) A is T-faithfully quadratic.

Proof. a) Let Y^*_P be the compact Hausdorff space of closed points in $\mathrm{Sper}(A, P)$. By Theorem 9.4 and Proposition 9.6, $\langle A, P \rangle$ is *strictly* representable in $R := \mathbb{C}(Y^*_P)$, that is, there is a ring morphism

$$\varphi : A \longrightarrow \mathbb{C}(Y^*_P),$$

satisfying conditions (1) − (3) in Definition 9.2. As above, whenever convenient we write $\breve{a}$ for $\varphi(a)$, where $a \in A$. Let

$$\text{(I)} \qquad \overline{T} \;=\; \{\textstyle\sum_{j=1}^n f_j^2\ \breve{b}_j : n \geq 1,\ f_j \in R \text{ and } b_j \in T,\ j \in \underline{\boldsymbol{n}}\}$$

be the preorder generated by $\breve{T} = \varphi[T]$ in R. Hence, for all $\alpha \in Y^*_{\overline{T}} \subseteq \mathrm{Sper}(R, \overline{T})$,

$$\text{(II)} \qquad T \;\subseteq\; \varphi^{-1}[\alpha] \in Y_T = \mathrm{Sper}(A, T).$$

To show T is unit-reflecting, suppose $x \in A^\times \cap \bigcap_{\beta \in Y_T} \beta \setminus (-\beta)$; then (II) entails $x \in A^\times \cap \bigcap_{\alpha \in Y^*_{\overline{T}}} \varphi^{-1}[\alpha \setminus (-\alpha)]$ and so $\breve{x} \in R^\times \cap \bigcap_{\alpha \in Y^*_{\overline{T}}} \alpha \setminus (-\alpha)$. By (the proof of) Theorem 8.29, $\overline{T}$, being a preorder of $\mathbb{C}(Y^*_P)$, is unit-reflecting, whence we obtain $\breve{x} \in \overline{T}$, i.e., there are $b_1, \ldots, b_n \in T$ and $h_1, \ldots, h_n \in R$ such that (see (I) above)

(III) $$\breve{x} \;=\; \sum_{j=1}^{n} h_j^2\ \breve{b}_j.$$

By removing from this sum all terms such that $\|\breve{b}_k\|_\infty = 0$, we may assume $\|b_j\|_\infty > 0$, for all $j \in \underline{\boldsymbol{n}}$.

Since $\breve{x} \in R^\times$, we have $m = \min\{|\breve{x}(p)| : p \in Y_P^*\} > 0$. Continuity of the product and denseness of $\varphi[A]$ in R (cf. 9.4.(1)) guarantee that for each $j \in \underline{\boldsymbol{n}}$, there is $c_j \in A$ such that

(IV) $$\|h_j^2 - \breve{c}_j{}^2\|_\infty \;<\; \frac{m}{2n(1+\|\breve{b}_j\|_\infty)}.$$

As in the proof of Fact 9.10, let

(*) $$z \;=\; c_1^2 b_1 + \ldots + c_n^2 b_n.$$

Clearly, z is in T. Since $\breve{z} = \sum_{i=1}^{n} \breve{c}_i{}^2 \breve{b}_i$, (III) and (IV) entail,

$$\|\breve{x} - \breve{z}\|_\infty \;<\; \frac{m}{2},$$

which in turn yields $[\![(\breve{xz}) > 0]\!] \;=\; Y_P^*$ (cf. proof of (IV) in 9.10). This and property (3) in 9.2 guarantee $xz \in P^\times \subseteq T^\times$. Since $z \in T$, we obtain $z \in T^\times$, which in turn yields $x = \dfrac{xz}{z} \in T^\times$, completing the proof of (a).

b) With notation as in Remark 9.3, let $\varphi : A \longrightarrow \mathbb{C}(X_P)$ be the strict representation of $\langle A, P \rangle$ given by Theorems 9.4 and 9.7. Recall (9.6) that X_T and Y_T^* are homeomorphic. Note that

(**) $$X_T \text{ is a closed set in } X_P.$$

Indeed, by the description of X_P in 9.5(1), if $h \in X_P \setminus X_T$, there is $t \in T$ such that $h(t) < 0$ in $\mathbb{R}$. Since X_P has the weak topology determined by the evaluation maps in $\breve{A}$, $[\![\breve{t} < 0]\!]$ is open in X_P, disjoint from X_T and $h \in [\![\breve{t} < 0]\!]$, establishing (**).

Let $P_{X_T} \;=\; \{f \in \mathbb{C}(X_P) : X_T \subseteq [\![f \geq 0]\!]\}$. As in 8.28, P_{X_T} is a proper u.r.-preorder of $\mathbb{C}(X_P)$. We claim that

(V) $$\breve{T} \;\subseteq\; P_{X_T}; \quad \text{and} \quad \text{for } a \in A^\times,\ \breve{a} \in P_{X_T} \;\Rightarrow\; a \in T^\times.$$

If $p \in T$ and $h \in X_T$, the definition of X_T yields $\breve{p}(h) = h(p) \geq 0$ in $\mathbb{R}$, i.e., $X_T \subseteq [\![\breve{p} \geq 0]\!]$.

Next, fix $a \in A^\times$ such that $X_T \subseteq [\![\breve{a} \geq 0]\!]$. Then, $\breve{a}$ is a unit in $\mathbb{C}(X_P)$ and so $X_T \subseteq [\![\breve{a} \geq 0]\!] = [\![\breve{a} > 0]\!]$. Now, the natural homeomorphism, $\mathfrak{h}$, between X_T and Y_T^* and item (3) in 9.2, yield

(VI) $$a \in A^\times \cap \bigcap_{h \in X_T} h^{-1}[\mathbb{R}^2 \setminus \{0\}] \;=\; A^\times \cap \bigcap_{\beta \in Y_T^*} \beta \setminus (-\beta).$$

Since T is unit-reflecting, Lemma 8.5 and (VI) guarantee $a \in T^\times$, completing the proof of (V).

By Theorems 3.9.(b) and 9.9 it suffices to prove that value representation in $\langle A, T \rangle$ by forms of arbitrary dimension is transversal.

Let $\psi = \langle b_1, \ldots, b_n \rangle$ be a form over $A^\times$ and let $a \in A^\times$ be such that $a \in D^v_{A,T}(\psi)$. Hence, $a = \sum_{j=1}^n s_j b_j$ with $s_j \in T$ ($j \in \underline{\boldsymbol{n}}$), which yields

$$\breve{a} \ = \ \textstyle\sum_{j=1}^n \breve{s}_j \, \breve{b}_j.$$

The first relation in (V) yields $\breve{s}_j \in P_{X_T}$, $j \in \underline{\boldsymbol{n}}$. Hence, $\breve{a} \in D^v_{X_T,\, P_{X_T}}(\breve{\psi})$. Now Theorem 8.21 guarantees that $\mathbb{C}(X_T)$ is P_{X_T}-faithfully quadratic and so there are $f_1, \ldots, f_n \in P^\times_{X_T}$ such that

$$\breve{a} \ = \ f_1 \breve{b}_1 + \cdots + f_n \breve{b}_n.$$

As in the proof of (a) and of Fact 9.10, we have, for $j \in \underline{\boldsymbol{n}}$:

- $m_j \ = \ \min\{f_j(p) \in \mathbb{R} : p \in X_P\} \ > \ 0$ and $X_T \subseteq [\![f_j > 0]\!]$;
- $\|\breve{b}_j\|_\infty > 0$;
- $m \ = \ \min\{|\breve{a}(p)| \in \mathbb{R} : p \in X_P\} \ > \ 0.$

Let $\varepsilon \ = \ \min\{m, m_1, \ldots, m_n\}$. Since $\breve{A}$ is a dense subalgebra of $\mathbb{C}(X_T)$ in the topology of uniform convergence,

For $j \in \underline{\boldsymbol{n}}$, there is $c_j \in A$ such that $\|\breve{c}_j - f_j\|_\infty \ < \ \dfrac{\varepsilon}{2n(1 + \|\breve{b}_j\|_\infty)}$.

The argument establishing (a) and ***(i)*** in the proof of 9.10 shows:

(VII) For each $j \in \underline{\boldsymbol{n}}$, $\breve{c}_j \in T^\times$ and $X_P \subseteq [\![\breve{c}_j > 0]\!]$.

Therefore, $\breve{c}_j \in P^\times_{X_P}$, and the second relation in (VII) guarantees that $c_j \in P^\times$, for all $j \in \underline{\boldsymbol{n}}$.

Let $z \ = \ \sum_{i=1}^n c_i b_i$. Exactly as in the proof of (a) and of ***(ii)*** in 9.10, we obtain $az \in P^\times \subseteq T^\times$, which yields $a = \dfrac{az}{z} \in D^t_{A,T}(\psi)$, completing the proof. ∎

CHAPTER 10

Quadratic Form Theory over Faithfully Quadratic Rings

The results in this Chapter exemplify how information concerning the mod 2 algebraic K-theory, the graded Witt ring and the theory of diagonal quadratic forms with unit coefficients in interesting classes of faithfully quadratic rings may be obtained from the theory of special groups.

10.1. **The Witt ring of T-faithfully quadratic p-rings.** Let $\langle A, T \rangle$ be a p-ring, where $T = A^2$ or is a preorder of A. Let S be a T-faithfully quadratic T-subgroup of A. If φ, ψ are forms over S, we say that they are **Witt equivalent mod T**, written $\varphi \sim_T^S \psi$, if there are integers n, m such that $\varphi \oplus m\langle 1, -1 \rangle \approx_T^S \psi \oplus n\langle 1, -1 \rangle$. Since S is T-faithfully quadratic, T-isometry is faithfully reflected by isometry in the special group $G_T(S)$, and so, by Proposition 1.6.(b), p. 4, of [**DM2**], Witt-cancellation holds for $\approx_T^S$. Hence, the set of equivalence classes $\overline{\varphi}$, of forms φ over S with respect to $\sim_T^S$,

$$W_T(S) \;=\; \{\overline{\varphi} : \varphi \text{ is a form over } S\},$$

with the operations of sum and product of classes induced by $\oplus$ and $\otimes$, is a commutative ring with identity $\overline{\langle 1 \rangle}$, whose zero is the class of any hyperbolic form. In fact, it is straightforward that $W_T(S)$ is naturally isomorphic to $W(G_T(S))$, the Witt ring of the special group $G_T(S)$, via the map induced on Witt rings by $\varphi \longmapsto \varphi^T$. Hence, $W_T(S)$ has all the properties described in paragraph 1.25 and Fact 1.26 (pp. 19-20) of [**DM2**]. In particular,

- $I_T(S) = I(G_T(S))$ is the *fundamental ideal of* $W_T(S)$, consisting of the classes of even dimensional forms;
- For $n \geq 1$, $I_T^n(S) = I^n(G_T(S))$, the n^{th}-power of $I_T(S)$, consists of all linear combinations, with coefficients in S, of Pfister forms of degree n over S;
- $\mathcal{W}_g^T(S) \;=\; \langle \mathbb{F}_2, \ldots, \overline{I_T^n}(S), \ldots \rangle$ is the **graded Witt ring** of S, where for $n \geq 1$,

$$\overline{I_T^n}(S) \;=\; I_T^n(S)/I_T^{n+1}(S).$$

When $T = A^2$, we omit T from the notation; moreover, if $S = A^\times$, Witt-equivalence mod T will be written $\sim_T$.

We recall (cf. [**DM5**], Definition 2.1):

- If φ is a form over S, the **Pfister index of degree n** of φ, $I(n, \varphi, S, T)$, is the least integer k such that φ is Witt-equivalent to a linear combination of k Pfister forms of degree n, if $\varphi \in I_T^n(S)$, and 0 otherwise;
- For each integer $m \geq 1$, the **m-Pfister index of S in degree n** is

$$I(n, m, S, T) \;=\; \sup\{I(n, \varphi, S, T) : \varphi \text{ a } m\text{-form over } S\} \in \mathbb{N} \cup \{\infty\}.$$

• Similarly one defines the notions of Pfister index of a form φ over a special group G, $I(n, m, \varphi, G)$, and m-Pfister index of G in degree n, $I(n, m, G)$. ■

Our first result is a description of both the mod 2 K-theory and of the graded Witt ring of Pythagorean ΣFAs and of Pythagorean rings that are Archimedean and have weak bounded inversion, showing, in particular, that they satisfy Milnor's mod 2 Witt ring conjecture. Examples of the first class are WRCRs, real closed rings and rings of continuous real-valued functions on a topological space; the second class includes, besides rings of continuous real-valued functions on a compact Hausdorff space, the real holomorphy ring of any formally real Pythagorean field.

THEOREM 10.2. *With notation as in* 10.1*, let A be a* Pythagorean ring. *Let Y^* be the subspace of closed points of* Sper(A). *Let $B(A)$ be the Boolean algebra of idempotents of A, and $B(Y^*)$ be the Boolean algebra of clopens in Y^*. Then,*

a) If A is an f-ring, then for all $n \geq 1$,

$$k_n A \;\simeq\; B(A) \;\simeq\; \overline{I^n}(A).$$

b) If A is Archimedean and has weak bounded inversion, then for all $n \geq 1$, $k_n A \simeq B(Y^*) \simeq \overline{I^n}(A)$.

REMARK 10.3. With notation as in 2.11, a special group G is called [SMC] if multiplication by $\lambda(-1) : k_n G \longrightarrow k_{n+1} G$ is injective for all $n \geq 0$; [SMC] special groups are necessarily reduced (see [**DM3**], [**DM7**]). ■

PROOF. a) By Theorem 8.20, A is faithfully quadratic and $G(A) = B(A)$. By Theorem 2.9 in [**DM7**], every Boolean algebra is a [SMC] special group and $k_n B = B$, for every $n \geq 1$. Moreover, by Theorem 4.1 and Corollary 4.2 in [**DM3**], $k_n B$ and $\overline{I^n}(B)$ are isomorphic. Since $k_* A$ is naturally isomorphic to $k_* G(A)$ (by Theorem 2.16), the discussion in 10.1 entails, for each $n \geq 1$, $k_n A \simeq k_n G(A) \simeq B(A) \simeq \overline{I^n}(G(A)) \simeq \overline{I^n}(A)$, as claimed.

b) By implication (2) ⇒ (1) in Theorem 9.7 and Proposition 9.6, A is strictly representable in $\mathbb{C}(Y^*)$, while Fact 9.11.(b.2) ensures $G(A)$ and $B(Y^*)$ to be isomorphic. Now, the proof is completed as in (a). ■

THEOREM 10.4. [Arason-Pfister Hauptsatz] *Let $\langle A, T \rangle$ be a proper p-ring and let S be a T-faithfully quadratic T-subgroup of A. If φ is a form over S such that* $\dim(\varphi) < 2^n$ *and $\varphi \in I^n_T(S)$, then φ is T-hyperbolic. In particular, $\bigcap_{n \geq 1} I^n_T(S) = \{0\}$.*

PROOF. By items (b) and (c) of Theorem 3.6, $G_T(S)$ is a *reduced* special group, faithfully representing T-isometry and value representation of diagonal quadratic forms with entries in S, while Theorem 7.31, p. 171 of [**DM2**], guarantees that $G_T(S)$ satisfies the Arason-Pfister Hauptsatz. The desired conclusion now follows from the T-quadratic faithfulness of S. ■

THEOREM 10.5. *Let $\langle A, T \rangle$ be a p-ring and T is a preorder on A, so that either:*

i) A is an f-ring and T a preorder containing the natural partial order $T_\sharp = {T_\sharp}^A$ of A, or

ii) T contains a preorder P such that $\langle A, P \rangle$ is Archimedean with bounded inversion.

Then,

a) The Arason-Pfister Hauptsatz holds in $\langle A, T \rangle$.

b) For all $a, b \in A^\times$, $\langle 1, a, b, -ab \rangle$ *is* T*-isotropic* (cf. 2.20.(a)).

c) With notation as in 10.1*, for all* $n, m \geq 1$,

$$I(n, m, A, T) \;=\; \begin{cases} \max\{1, m/2^n\} & \textit{if } m \textit{ is even;} \\ 0 & \textit{if } m \textit{ is odd.} \end{cases}$$

Hence, for each $m, n \geq 1$*, the Pfister index* $I(n, m, A, T)$ *(cf.* 10.1) *is uniformly bounded in the class of rings of either type* (i) *or* (ii).

d) Let $\langle R, P \rangle$ *be another ring of the same type as* $\langle A, T \rangle$*. If* $\langle A, T \rangle \xrightarrow{f} \langle R, P \rangle$ *is a p-ring morphism, the following are equivalent, where* $f \star \langle a_1, \ldots, a_n \rangle = \langle f(a_1), \ldots, f(a_n) \rangle$:

(1) *f is complete, that is, for all n-forms φ, ψ over $A^\times$,*

$$\varphi \approx_T \psi \;\Leftrightarrow\; f \star \varphi \approx_P f \star \psi;$$

(2) $f^{-1}[P^\times] \;=\; T^\times$;

(3) *f reflects isotropy, i.e., if φ is a form over $A^\times$ and $f \star \varphi$ is P-isotropic, then φ is T-isotropic* (cf. 2.20.(a)).

These conditions are, in turn, equivalent to

(4) *The induced π-SG morphism, $f^\pi : G_T(A) \longrightarrow G_P(R)$, is injective.*

Proof. To avoid repetition of similar arguments, we do the proof in case $\langle A, T \rangle$ is of type (i).

By Theorem 8.21, $\langle A, T \rangle$ is faithfully quadratic and $G_T(A)$ is isomorphic to $B = B(A)/\mathcal{I}$, where $B(A)$ is the Boolean algebra of idempotents in A and $\mathcal{I}$ is the ideal $\{e \in B(A) : H(e) \cap Y_T = \emptyset\}$, generated by $Y_T = \mathrm{Sper}(A, T)$. We shall employ the observations in 10.1 without further comment. In particular, item (a) is an immediate consequence of Theorem 10.4.

b) By Proposition 7.17, p. 153, of [**DM2**], a reduced special group is a Boolean algebra iff every form of the type $\langle 1, x, y, -xy \rangle$ is isotropic; since $\langle A, T \rangle$ is T-faithfully quadratic, the desired conclusion follows from the fact that $G_T(A)$ is a Boolean algebra.

c) By Proposition 3.17, p. 231, of [**DM5**], for every $n, m \geq 1$ and every Boolean algebra B (considered as a reduced special group),

$$I(n, m, B) \;=\; \begin{cases} \max\{1, m/2^n\} & \text{if } m \text{ is even;} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Since $G_T(A)$ is itself a Boolean algebra, the conclusion follows.

d) We start by noting the following

Fact 10.6. *Let $g : \langle A_1, T_1 \rangle \xrightarrow{g} \langle A_2, T_2 \rangle$ be a p-ring morphism and let $g^\pi : G_{T_1}(A_1) \longrightarrow G_{T_2}(A_2)$ be the induced π-SG morphism, $g^\pi(a^{T_1}) = g(a)^{T_2}$, as in* 2.8*. Then, g^π is injective iff $g^{-1}[T_2^\times] = T_1^\times$.*

Proof. Since g^π is a group morphism, it will be injective iff $\ker g^\pi = \{1\}$. Since the kernel of the map $a \in A_i^\times \longmapsto a^{T_i} \in G_{T_i}(A_i)$ is $T_i^\times$, $i = 1, 2$, and $g[T_1] \subseteq T_2$, the stated equivalence is clear. ■

Note that this Fact yields at once (2) $\Leftrightarrow$ (4).

<u>(1) $\Rightarrow$ (2) :</u> As in Fact 10.6, let $f^{\pi} : G_T(A) \longrightarrow G_P(R)$ be the induced π-SG morphism. Since A is T-faithfully quadratic and R is P-faithfully quadratic (Theorem 8.21), f^{π} is, in fact, a morphism of reduced special groups. Moreover, note that for all forms $\theta = \langle a_1, \ldots, a_n \rangle$ over $A^{\times}$,

$$(f \star \theta)^P = \langle f(a_1)^P, \ldots, f(a_1)^P \rangle = \langle f^{\pi}(a_1^T), \ldots, f^{\pi}(a_n^T) \rangle = f^{\pi} \star \theta^T.$$

Hence, if φ, ψ are n-forms over $A^{\times}$, assumption (1) and quadratic faithfulness entail

$$\varphi^T \equiv_T \psi^T \quad \Leftrightarrow \quad f^{\pi} \star \varphi^T \equiv_P f^{\pi} \star \psi^T.$$

Hence, since $G_T(A)$ is reduced, Remark 1.15 guarantees that f^{π} is a *complete embedding* of *reduced* special groups. In particular, f^{π} is injective and Fact 10.6 entails (2).

<u>(2) $\Rightarrow$ (3) :</u> By Fact 10.6, $f^{\pi} : G_T(A) \longrightarrow G_P(R)$ is injective. Since $G_T(A)$ and $G_P(R)$ are Boolean algebras, Example 5.38, p. 98, of [**DM2**] implies that f^{π} is *isotropy-reflecting*. But then, quadratic faithfulness and the argument used in the proof of (1) $\Rightarrow$ (2) yields (3).

<u>(3) $\Rightarrow$ (1) :</u> It is shown in Proposition 5.3.2, p. 96, of [**DM2**], that any isotropy-reflecting SG-morphism is a complete embedding [1]. Since (3) is equivalent to f^{π} being isotropy-reflecting (as pointed out in (2) $\Rightarrow$ (3)) and (1) is equivalent to f^{π} being a complete embedding (cf. proof of (1) $\Rightarrow$ (2)), the desired conclusion follows, ending the proof. ■

Recall that if $\langle A, T \rangle$ is a p-ring, we write Y_T^* for the compact Hausdorff space of closed points in Sper(A, T).

THEOREM 10.7. [Marshall's signature conjecture; refined version]

a) Let $\langle A, P \rangle$ be an Archimedean p-ring with bounded inversion and let T be a preorder containing P. Let φ be a form over $A^{\times}$. With notation as in Definition 3.14*, if for some dense subset $D \subseteq Y_T^*$, we have*

$$\textit{For all } \beta \in D, \quad sgn_{\tau_\beta}(\varphi) \equiv 0 \bmod 2^n,$$

then $\varphi \in I_T^n(A)$.

b) Let A be an f-ring and let $T_\sharp$ be its natural partial order. Let T be a preorder on A, such that $T_\sharp \subseteq T$. Let φ be a form over $A^{\times}$. If for some dense subset D of Y_T^* we have

$$\textit{For all } \beta \in D, \quad sgn_{\tau_\beta}(\varphi) \equiv 0 \bmod 2^n,$$

then $\varphi \in I_T^n(A)$.

PROOF. The argument that follows applies to both (a) and (b). Recall that $Z_{A,T}$ and $X_{G_T(A)}$ are the space of signatures of $\langle A, T \rangle$ and the space of characters of $G_T(A)$, respectively (cf. Definitions 3.14 and 1.12.(b)). We first observe:

FACT 10.8. *If the congruence* mod 2^n *above holds on a dense subset of Y_T^*, then it holds on all of Y_T^*.*

PROOF. If $\varphi = \langle a_1, \ldots, a_m \rangle$, given $\alpha \in Y_T^*$, with $\tau_\alpha \in Z_{A,T}$ as in 3.15, let

[1] In general, the converse fails.

$$V(\varphi, \alpha) = \bigcap_{j=1}^{m} [\![\overline{a}_j = \tau_\alpha(a_j)]\!];$$

then $V(\varphi, \alpha)$ is a non-empty clopen in Y_T^* ($\alpha \in V(\varphi, \alpha)$) and so, there is $\beta \in D \cap V(\varphi, \alpha)$. But then, $\tau_\beta(a_j) = \tau_\alpha(a_j)$ for $1 \leq j \leq m$, whence $sgn_{\tau_\alpha}(\varphi) = sgn_{\tau_\beta}(\varphi) \equiv 0 \bmod 2^n$, as desired. □

But then, Fact 10.8 and Lemma 3.17.(b) imply that the hypothesis on φ is equivalent to:

$$\text{For all } \tau \in Z_{A,T}, \quad sgn_\tau(\varphi) \equiv 0 \bmod 2^n,$$

whence Lemma 3.16 entails

(*) For all SG-characters σ of $G_T(A)$, $sgn_\sigma(\varphi^T) \equiv 0 \bmod 2^n$.

Now, recall that:

• $G_T(A)$ is a Boolean algebra (by Fact 9.11.(b.2) in case (a) and by Theorem 8.21 in case (b)), and

• Boolean algebras are [SMC]-special groups (Theorem 2.9, p. 203 of [**DM7**]) and hence – by Proposition 4.4, p. 168 of [**DM3**] or Lemma 1.2 of [**DM8**] – verify Marshall's signature conjecture.

From (*) and the preceding observations we obtain $\varphi^T \in I^n(G_T(A))$. By the T-quadratic faithfulness of A, this is equivalent to $\varphi \in I_T^n(A)$, proving Theorem 10.7. ■

Remark 10.9. The denseness assertions in Proposition 3.18.(3) and Theorem 10.7.(b) provide a tool that, in many cases, simplifies testing the isometry of forms and the membership in I^n, respectively. This observation is illustrated in case of the ring $A = \mathbb{C}(X)$.

Since the topological space X itself is dense in the compact space Y^* of closed points of Sper(A) (4.13.(c)), the above mentioned assertions reduce the testing of those properties to computations of signs at the points of X, dispensing with the need to consider the totality of all orderings in Y^* ($= \beta X$, in general an ill-understood space).

In fact, for $f \in \mathbb{C}(X)^\times$ and $x \in X$, we have

$$\tau_{\alpha_x}(f) \;=\; sgn(f(x)) \;=\; \frac{f(x)}{|f(x)|},$$

where α_x is the ordering associated to the point $x \in X$. Thus, if $\varphi = \langle f_1, \ldots, f_n \rangle$ is a form over $A^\times$,

$$sgn_{\tau_{\alpha_x}}(\varphi) \;=\; \sum_{i=1}^{n} \frac{f_i(x)}{|f_i(x)|},$$

and 10.7.(b) gives

(I) $$\varphi \in I^n(\mathbb{C}(X)) \quad \Leftrightarrow \quad \forall\, x \in X,\; \sum_{i=1}^{n} \frac{f_i(x)}{|f_i(x)|} \;\equiv\; 0 \bmod 2^n.$$

Likewise, if $\psi = \langle g_1, \ldots, g_n \rangle$ is another such form, 3.18.(3) shows

(II) $$\varphi \approx \psi \quad \Leftrightarrow \quad \forall\, x \in X,\; \sum_{i=1}^{n} \frac{f_i(x)}{|f_i(x)|} \;=\; \sum_{i=1}^{n} \frac{g_i(x)}{|g_i(x)|}.$$

Since a unit $f \in \mathbb{C}(X)^\times$ does not change sign on a connected component of X, to verify membership in $I^n(\mathbb{C}(X))$ or isometry of forms, it suffices to check the validity of the right-hand side of (I) or (II), respectively, at just one point of each connected component of X. ■

THEOREM 10.10. *Let A be an f-ring and let $T_\sharp$ be its natural partial order. For n-forms $\varphi = \langle a_1, \ldots, a_n \rangle$ and $\psi = \langle b_1, \ldots, b_n \rangle$ over $A^\times$, the following are equivalent:*

(1) $\varphi \approx_{T_\sharp} \psi$;

(2) *For each $1 \le k \le n$, let $S^{n,k}$ be the set of all strictly increasing sequences, $p = (p_1, \ldots, p_k)$, of length k of elements of $\underline{\boldsymbol{n}}$. Then, for all $k \in \underline{\boldsymbol{n}}$,*

$$\mathcal{HT}_k(\varphi) \;=\; \bigvee_{p \in S^{n,k}} \prod_{i=1}^{k} \mathfrak{e}^-(a_{p_i}) \;=\; \bigvee_{p \in S^{n,k}} \prod_{i=1}^{k} \mathfrak{e}^-(b_{p_i}) \;=\; \mathcal{HT}_k(\psi).$$

(3) [Local-global Sylvester's inertia law] *There is an orthogonal decomposition of A into idempotents, $\{e_1, \ldots, e_m\}$, such that for every $1 \le j \le m$, the following conditions are satisfied:*

(*i*) *Each entry in φ and ψ is either in ${T_\sharp}^\times e_j$ (strictly positive in Ae_j), or in $-({T_\sharp}^\times e_j)$ (strictly negative in Ae_j), i.e.,*

$$\begin{aligned}\underline{\boldsymbol{n}} \;&=\; \{k \in \underline{\boldsymbol{n}} : a_k e_j >_{T_\sharp} 0\} \;\cup\; \{k \in \underline{\boldsymbol{n}} : a_k e_j <_{T_\sharp} 0\} \\ &=\; \{k \in \underline{\boldsymbol{n}} : b_k e_j >_{T_\sharp} 0\} \;\cup\; \{k \in \underline{\boldsymbol{n}} : b_k e_j <_{T_\sharp} 0\};\end{aligned}$$

(*ii*) *The number of entries of φ and ψ that are strictly negative in Ae_j is the same, i.e.*

$$\operatorname{card}(\{k \in \underline{\boldsymbol{n}} : a_k e_j <_{T_\sharp} 0\}) \;=\; \operatorname{card}(\{k \in \underline{\boldsymbol{n}} : b_k e_j <_{T_\sharp} 0\}).$$

Note. $\mathcal{HT}_k(\varphi)$ denotes the k^{th} Horn-Tarski invariant of the form φ ($1 \le k \le n$); cf. [**DM2**], Definition 7.2, p. 140. In this proof we shall freely use the theory of these invariants, developed in Chapter 7 of [**DM2**].

PROOF. Let $B = B(A)$ be the Boolean algebra of idempotents in A and let $G_\sharp = G_\sharp(A)$. By Theorem 8.20 (and with notation therein), the map $\mathfrak{b}_A$ given by $a^\sharp \in G_\sharp \longmapsto \mathfrak{e}^-(a) \in B$ is a SG-isomorphism, by which we may identify $G_\sharp$ with B. We also note that if e is a non-zero idempotent in A, then $T_\sharp e$ is a *proper* partial order on Ae and the *reduced* special group, $G_{T_\sharp e}(Ae)$, associated to $\langle Ae, T_\sharp e \rangle$, may be identified with Be.

Since meet in B is product, and A is $T_\sharp$-faithfully quadratic, the equivalence between (1) and (2) is a consequence of Theorem 7.1, p. 136, of [**DM2**] : it is easily checked that the $\mathcal{HT}_k(\varphi)$ in the statement of (2) are precisely the Horn-Tarski invariants of the (image of the) form φ in B. We note, *in passim*, that by Fact 4.18, p. 73, of [**DM2**], the Boolean hull of a Boolean algebra may be identified with itself.

To complete the proof it suffices to show that (3) and (2) are equivalent. We recall, as observed in (IV) of the proof of Theorem 8.21, that if f is an idempotent in A,

(*) For all $z \in A^\times$, in the ring Af,

$$\mathfrak{e}^+(zf) = \mathfrak{e}^+(z)f \quad \text{and} \quad \mathfrak{e}^-(zf) = \mathfrak{e}^-(z)f.$$

We start with

FACT 10.11. *Let $\theta = \langle c_1, \ldots, c_n \rangle$ be a form over $A^\times$ and let e be a non-zero idempotent in A such that*

$$\underline{\boldsymbol{n}} \;=\; \{\ell \in \underline{\boldsymbol{n}} : c_\ell\, e >_{T_\sharp} 0\} \;\cup\; \{\ell \in \underline{\boldsymbol{n}} : c_\ell\, e <_{T_\sharp} 0\}.$$

Let $N_\theta = \{\ell \in \underline{\boldsymbol{n}} : c_\ell\, e <_{T_\sharp} 0\}$. *Then, for all* $1 \le k \le n$,

$$\mathcal{HT}_k(\theta e) \;=\; \begin{cases} e & \text{if } k \le \mathrm{card}(N_\theta); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\ell \in N_\theta$, then, recalling Theorem 8.13.(a.3) and using (*) above, we have

$$c_\ell\, e \;=\; |c_\ell|(\mathfrak{e}^+(c_\ell) - \mathfrak{e}^-(c_\ell))\, e \;=\; |c_\ell\, e|(\mathfrak{e}^+(c_\ell\, e) - \mathfrak{e}^-(c_\ell\, e)) \;<_{T_\sharp}\; 0.$$

Then, the positive part of $c_\ell e$ must be 0, whence, because $|c_\ell| \in A^\times$, we obtain $\mathfrak{e}^+(c_\ell\, e) = \mathfrak{e}^+(c_\ell)\, e = 0$, i.e. $\mathfrak{e}^-(c_\ell\, e) = \mathfrak{e}^-(c_\ell)\, e = e$. Similarly, we show that $\ell \notin N_\theta$ implies $\mathfrak{e}^-(c_\ell\, e) = \mathfrak{e}^-(c_\ell)\, e = 0$. Hence,

$$\text{(I)} \qquad \begin{cases} \forall\, \ell \in N_\theta, \;\; \mathfrak{e}^-(c_\ell\, e) = \mathfrak{e}^-(c_\ell)\, e = e \quad \text{and} \\ \forall\, \ell \notin N_\theta, \;\; \mathfrak{e}^-(c_\ell\, e) = \mathfrak{e}^-(c_\ell)\, e = 0. \end{cases}$$

For $k \in \underline{\boldsymbol{n}}$, the k^{th} Horn-Tarski invariant of θe **in** $\boldsymbol{Be} \subseteq B$ is

$$\text{(II)} \;\; \mathcal{HT}_k(\theta e) \;=\; \bigvee_{p \in S^{n,k}} \prod_{i=1}^k \mathfrak{e}^-(c_{p_i} e) \;=\; \bigvee_{p \in S^{n,k}} \prod_{i=1}^k \mathfrak{e}^-(c_{p_i}) e \;=\; \mathcal{HT}_k(\theta) e.$$

If $k \le \mathrm{card}(N_\theta)$, then there is $p \in S^{n,k}$ such that $p \subseteq N_\theta$; hence, (I) entails $\prod_{i=1}^k \mathfrak{e}^-(c_{p_i})e = e$ and so (since all terms in (II) are $\le e$), $\mathcal{HT}_k(\theta e) = e$. If $k > \mathrm{card}(N_\theta)$, every p in $S^{n,k}$ has non-empty intersection with the complement of N_θ, whence (I) and (II) imply $\mathcal{HT}_k(\theta e) = 0$, as asserted. □

<u>(2) ⇒ (3) :</u> By Theorem 8.13.(e), there is an orthogonal decomposition of A into (non-zero) idempotents, $\{e_1, \ldots, e_m\}$, subordinate to $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$, satisfying (1) and (2) in 8.13.(e). Note that 8.13.(e.2) immediately entails (3.i) (in the partial order $T_\sharp$, every non-zero idempotent is strictly positive). Fix $1 \le j \le m$ and set $e_j = e$. We claim that the coefficients of φ and ψ satisfy (3.ii) with respect to e. Let N_φ, N_ψ be as in the statement of Fact 10.11. From (II) in the proof of Fact 10.11 and assumption (2) we obtain, for all $1 \le k \le n$,

$$\text{(III)} \qquad \mathcal{HT}_k(\varphi e) \;=\; \mathcal{HT}_k(\varphi)e \;=\; \mathcal{HT}_k(\psi)e \;=\; \mathcal{HT}_k(\psi e).$$

Since φ and ψ both satisfy (3.i) with respect to $e \neq 0$, (III) and the description of the Horn-Tarski invariants in Fact 10.11 immediately imply $\mathrm{card}(N_\varphi) = \mathrm{card}(N_\psi)$, as claimed.

<u>(3) ⇒ (1) :</u> Let $\{e_1, \ldots, e_m\}$ be an orthogonal decomposition of A into non-zero idempotents, satisfying (3.i) and (3.ii). By Proposition 4.5.(d), to show that $\varphi \approx_{T_\sharp} \psi$ it is enough to check that $\varphi e_j \approx_{T_\sharp e_j} \psi e_j$, for all $1 \le j \le m$. By the observations at the beginning of the proof, we know that Be_j is the RSG associated to $\langle\, Ae_j, T_\sharp e_j \,\rangle$, as well as its Boolean hull.

It follows from Fact 10.11 that, if φ, ψ satisfy conditions (3.i) and (3.ii) with respect to e_j, then $\mathcal{HT}_k(\varphi e_j) = \mathcal{HT}_k(\psi e_j)$, $1 \le k \le n$. By Theorem 7.1, p. 136, of [**DM2**], we get $(\varphi e_j)^\sharp = \varphi^\sharp e_j$ isometric to $(\psi e_j)^\sharp = \psi^\sharp e_j$ in $G_{T_\sharp e_j}(Ae_j)$. But then, the $T_\sharp e_j$-quadratic faithfulness of Ae_j (Corollary 4.7) yields $\varphi e_j \approx_{T_\sharp e_j} \psi e_j$, as required. ■

Remarks 10.12. a) Let A be an f-ring and let $T_\sharp$ be its natural partial order. If A is a *domain*, then $T_\sharp$ is a linear order, and $A^\times = T_\sharp{}^\times \cup -T_\sharp{}^\times$. Hence, A has only one $T_\sharp$-subgroup, namely $A^\times$, and $G_\sharp(A) = \mathbb{Z}_2$, is the two-element Boolean algebra.

b) If $A = \mathbb{C}(X)$ is the ring of real-valued continuous functions on a completely regular space, X, the Boolean algebra of idempotents in A may be identified with $B(X)$, the Boolean algebra of clopens in X. With this identification, condition (3) in 10.10 states that, given forms φ, ψ of the same dimension over $A^\times$, there is clopen partition, $\mathcal{D}$, of X, such that none of the coefficients of φ and ψ change sign in any $V \in \mathcal{D}$, and the number of coefficients of φ and ψ that have a fixed sign on each $V \in \mathcal{D}$ is the same.

c) With appropriate modifications, the equivalence of conditions (1) and (2) in Theorem 10.10 is true for any Archimedean p-ring, $\langle A, T \rangle$, with bounded inversion. By Theorem 9.7, A is strictly representable in a compact Hausdorff space X, while Fact 9.11 in the proof of Theorem 9.9 shows that $G_T(A)$ is SG-isomorphic to $B(X)$, via the map $\overline{\gamma}$ given by $a^T \in G_T(A) \longmapsto [\![\breve{a} < 0]\!] \in B(X)$. Corresponding to the conditions (1) and (2) in 10.10, and with $S^{n,k}$ defined as in 10.10.(3), we have:

FACT 10.13. *If $\langle A, T \rangle$ is an Archimedean p-ring with bounded inversion and $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_n \rangle$ are forms over $A^\times$, the following are equivalent:*

(1) $\varphi \approx_T \psi$;

(2) *For each* $1 \leq k \leq n$, $\mathcal{HT}_k(\breve{\varphi}) = \bigcup_{p \in S^{n,k}} \bigcap_{i=1}^{n} [\![\breve{a_{p_i}} < 0]\!] = \bigcup_{p \in S^{n,k}} \bigcap_{i=1}^{n} [\![\breve{b_{p_i}} < 0]\!] = \mathcal{HT}_k(\breve{\psi})$. ■

Item (2) expresses the Horn-Tarski conditions for the isometry of $\overline{\gamma} \star \varphi$ and $\overline{\gamma} \star \psi$ in the Boolean algebra $B(X)$. ■

EXAMPLE 10.14. **A q-subgroup of a faithfully quadratic ring which is not faithfully quadratic.** Let $\langle A, T \rangle$ be a proper p-ring and let S be a T-subgroup of A. Then, $G_T(S) = S/T^\times$ is a π-SG subgroup of $G_T(A) = A^\times/T^\times$, and both are reduced. We begin by:

FACT 10.15. *a) If A satisfies* [T-FQ 1] *or* $[\text{T-FQ } 3]_3$ *the same is true of S.*

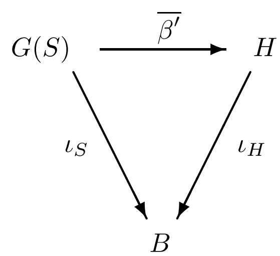
b) If A and S are T-faithfully quadratic, then $G_T(S)$ is a complete *subgroup of $G_T(A)$, that is, if φ, ψ are n-forms over S,*

$$\varphi^T \equiv^A_T \psi^T \;\; \Rightarrow \;\; \varphi^T \equiv^S_T \psi^T.$$

PROOF. a) The assertion about [T-FQ 1] is clear. For the other, suppose a, u, v, x, $y \in S$ are such that $\langle a, u, v \rangle \approx^S_T \langle a, x, y \rangle$; then, the same relation is true with $\approx^S_T$ replaced by $\approx^A_T$. Since A verifies $[\text{T-FQ } 3]_3$, we obtain $\langle u, v \rangle \approx^A_T \langle x, y \rangle$, which entails $\langle u, v \rangle \approx^S_T \langle x, y \rangle$ because $\approx^A_T$ (resp., $\approx^S_T$) coincide with $\equiv_{G_T(A)}$ (resp., $\equiv_{G_T(S)}$) on binary forms, and $G_T(S)$ is a π-SG subgroup of $G_T(A)$.

b) By Theorem 5.2, p. 75, of [**DM2**], it suffices to check that if $\mathcal{P}$ is a Pfister form with entries in S, then $\mathcal{P}^T$ hyperbolic in $G_T(A)$ entails $\mathcal{P}^T$ hyperbolic in $G_T(S)$. Since $G_T(A)$ is reduced, $\mathcal{P}^T$ is hyperbolic iff $-1 \in D_{G_T(A)}(\mathcal{P}^T)$; then, the T-quadratic faithfulness of $\langle A, T \rangle$ entails $-1 \in D^v_{A,T}(\mathcal{P})$ and so, the coefficients of $\mathcal{P}$ being in S, we obtain $-1 \in D^v_{S,T}(\mathcal{P})$. Since S is T-faithfully quadratic, we get $-1 \in D_{G_T(S)}(\mathcal{P}^T)$, and $\mathcal{P}^T$ is hyperbolic in $G_T(S)$, as needed. □

Let 2^ω be the Cantor space and let $B = B(2^\omega)$ be the Boolean algebra of clopens in 2^ω. By Example 5.10, p. 83ff, of [**DM2**], B has a *non-complete reduced special subgroup*, H; in fact, H may be taken to be a *finite fan*. By Proposition 8.25, $A = \mathbb{C}(2^\omega)$ is faithfully quadratic; moreover, the group morphism $f \in A \longmapsto \beta(f) = [\![f < 0]\!] \in B$, induces a natural isomorphism, $\beta_{2^\omega} : G(A) \longrightarrow B$, given by $\beta_{2^\omega}(\widehat{f}\,) = [\![f < 0]\!]$. Now, *the proof* of Theorem 2.3 in [**DM9**] furnishes a q-subgroup (i.e., a $A^2 = \Sigma A^2$-subgroup) S of A, such that $\beta' = \beta \upharpoonright S : S \longrightarrow H$ is onto and has kernel $A^{\times 2}$, yielding a SG-isomorphism, $\overline{\beta'} : G(S) \longrightarrow H$, making the following diagram commutative:



where ι_H is the inclusion and $\iota_S = \beta_{2^\omega} \upharpoonright G(S)$. Since $\overline{\beta'}$ is a SG-isomorphism and ι_H is not a complete embedding (H is not a complete subgroup of B), it follows that ι_S is not a complete embedding, i.e., $G(S)$ is not a complete subgroup of $B = G(A)$. By Fact 10.15.(b), S *is not faithfully quadratic*. Note that $G(S)$ is a reduced special group, and, by Fact 10.15.(a), S verifies [T-FQ 1] and $[\text{T-FQ } 3]_3$. ■

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